Finding the shortest path

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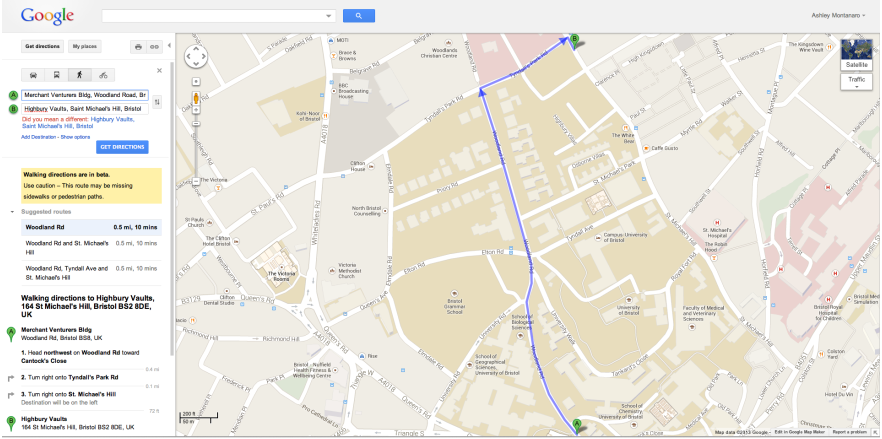
A fundamental task with many applications:



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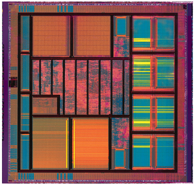
Other applications

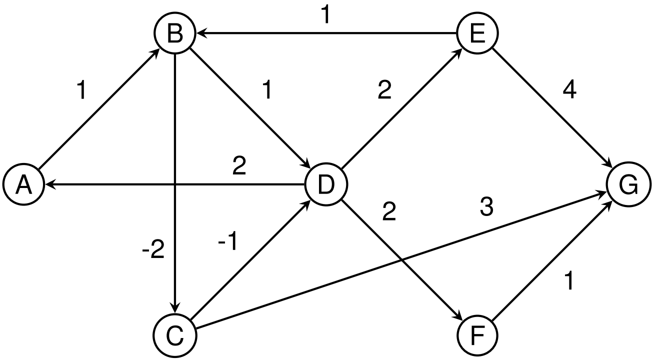
- Internet routing (e.g. the OSPF routing algorithm)
- VLSI routing
- Traffic information systems
- Robot motion planning
- Routing telephone calls
- Avoiding nuclear contamination
- Destabilising currency markets
- ...



Pics: Wikipedia, autoevolution.com, autoblog.com

Shortest paths problem

Formally, a shortest path from s to t in a graph G is a sequence $v_1, v_2, \dots, v_m$ such that the total weight of the edges $s \to v_1$, $v_1 \to v_2$, ..., $v_m \to t$ is minimal.



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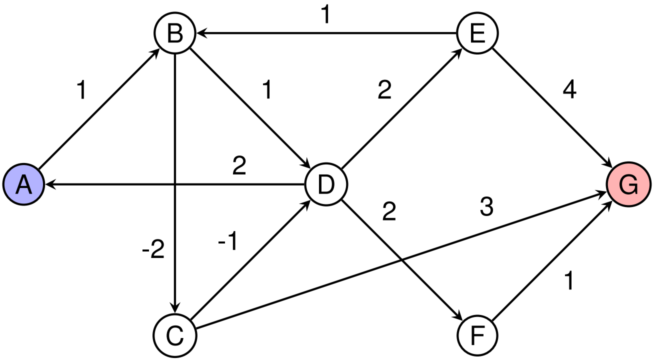

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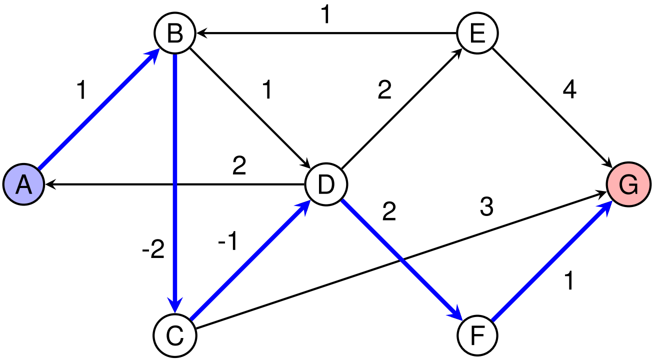

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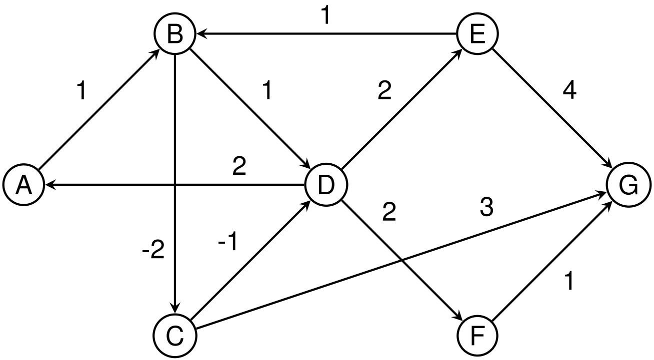

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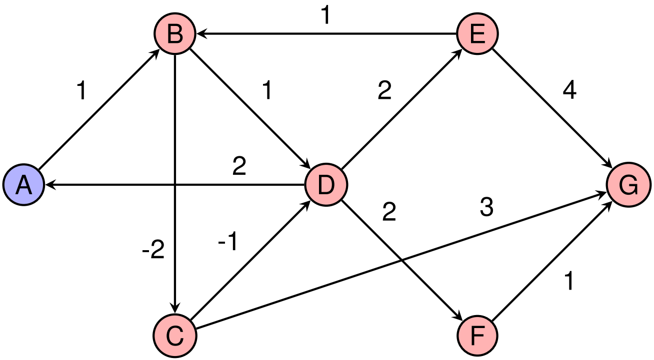

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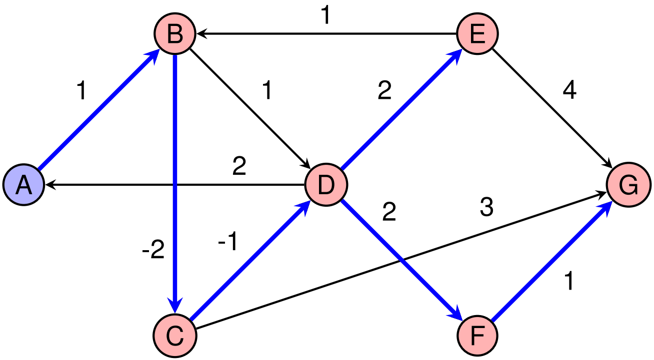

Negative-weight edges

- If some of the edges have negative weights, the idea of a shortest path might not make sense.
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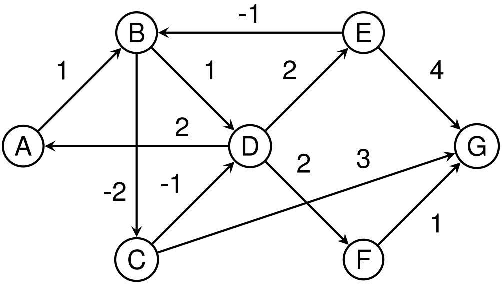

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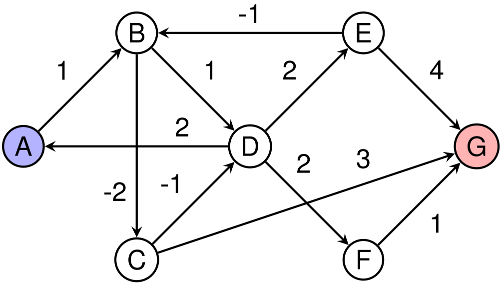

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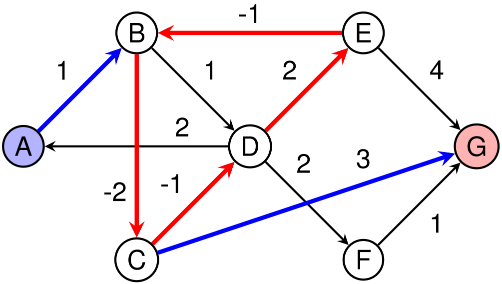

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- There can be exponentially many paths so such an algorithm cannot be efficient.

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- For each vertex v, we will maintain a guess for its distance from s; call this $v.d$.

Predecessors and shortest paths

- For each vertex v, we try to determine its predecessor $v.\pi$, which is the previous vertex in some shortest path from s to v.
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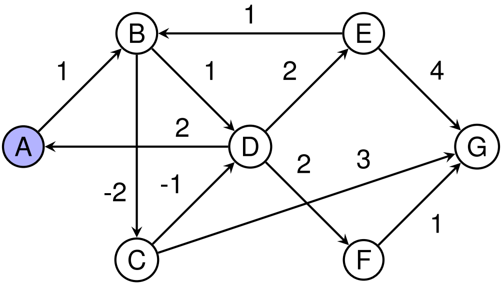

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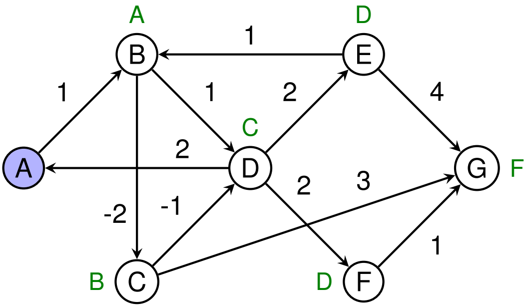

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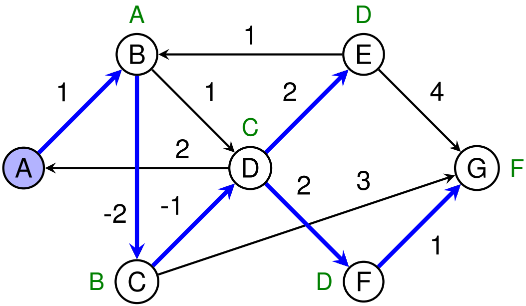

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2. Update our guesses by relaxing edges:

- If there is an edge $u \to v$ and our guess for the distance from s to v is greater than our guess for the distance from s to u, plus $w(u, v)$, then we can improve our guess by using this edge.

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Relax(u, v)

1. if $v.d > u.d + w(u, v)$
2. $\quad\quad v.d \leftarrow u.d + w(u, v)$
3. $\quad\quad v.\pi = u$

Note that $\infty + x = \infty$ for any real number x.

The Bellman-Ford algorithm

This algorithm simply consists of repeatedly relaxing every edge in G.

BellmanFord(G, s)

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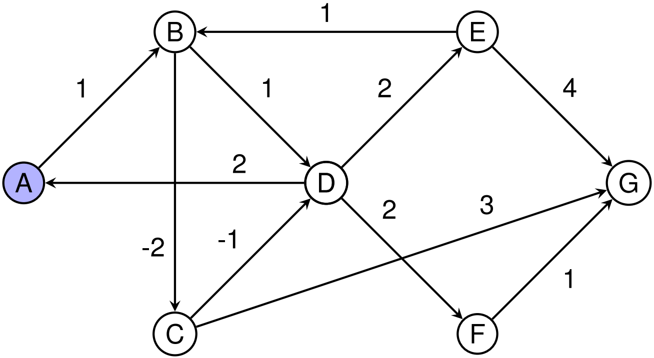
- Time complexity: $\Theta(V) + \Theta(VE) + \Theta(E) = \Theta(VE)$.

Example 1: no negative-weight cycles

Imagine we want to find shortest paths from vertex A in the following graph:



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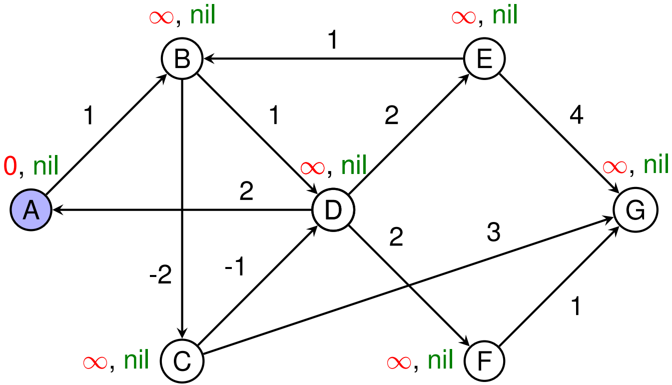

- In the above diagram, the red text is the distance from the source A, (i.e. $v.d$), and the green text is the predecessor vertex (i.e. $v.\pi$).

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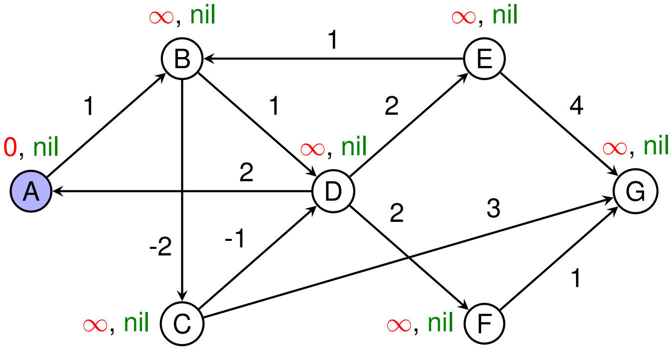
The first iteration of the for loop:



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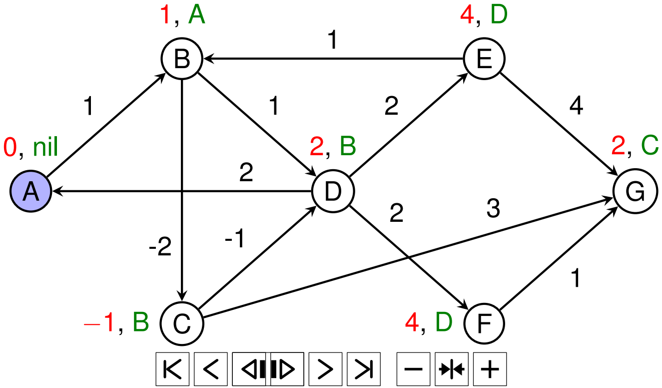

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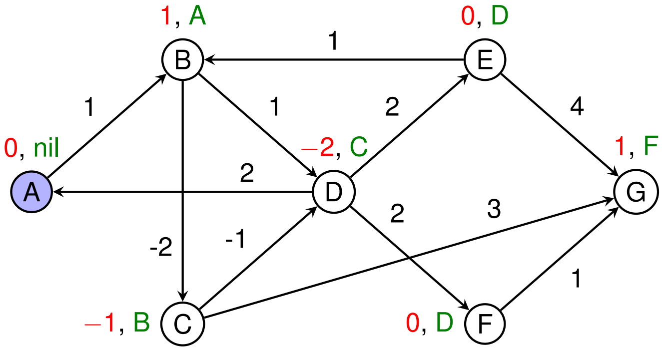

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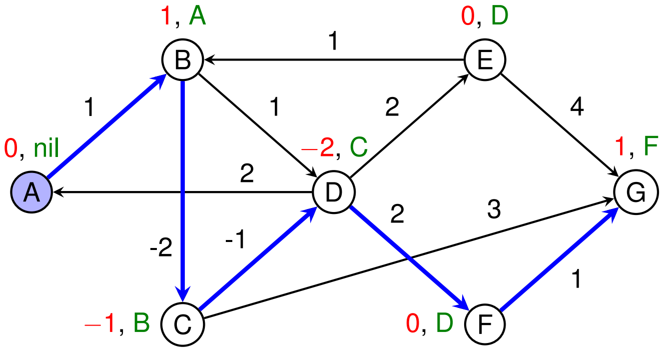

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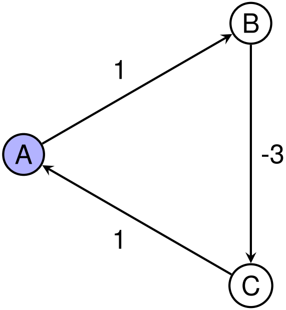
Example 2: negative-weight cycle

We now consider an input graph that has a negative-weight cycle.



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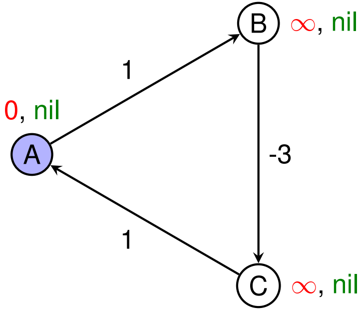

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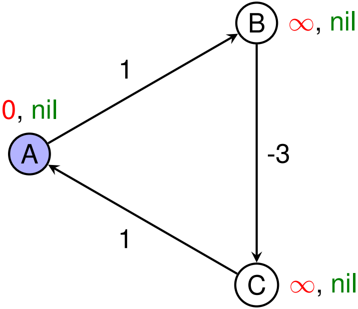

- As before, the order in which we consider the edges is arbitrary (here we use the order A $\rightarrow$ B, C $\rightarrow$ A, B $\rightarrow$ C).

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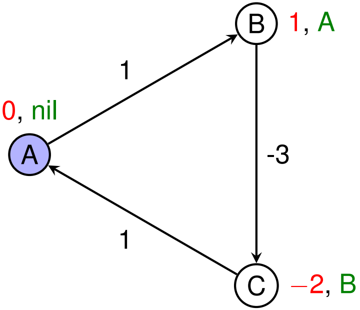

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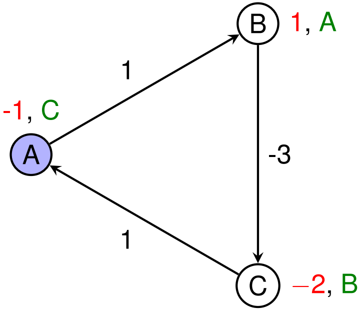

- At the end of the algorithm, $B.d > A.d + w(A, B)$.
- So the algorithm terminates with “Negative-weight cycle detected”.

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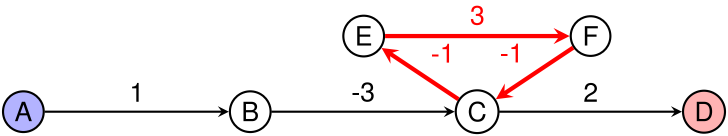

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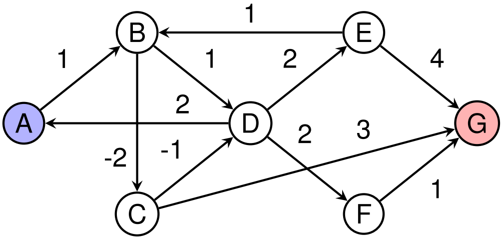

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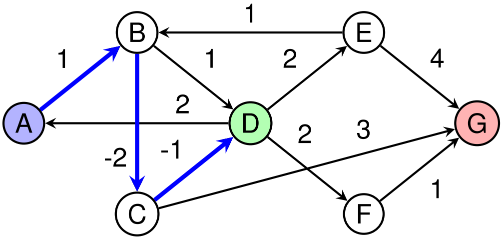

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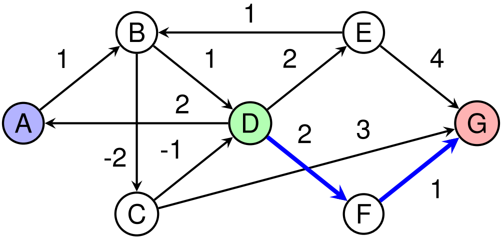

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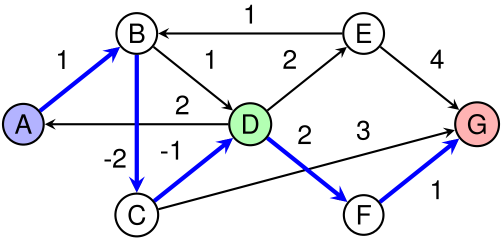

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Finally, an important property of relaxation, which can be proven by induction and using the triangle inequality, is called path-relaxation:

Claim (path-relaxation)

Assume that:

- $p = s \to v_1 \to \cdots \to v_k \to v$ is a shortest path from s to v;

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Proof: exercise.

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If G does not contain a negative-weight cycle reachable from s, then at the completion of BellmanFord, $v.d = \delta(s, v)$ for all vertices v.

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- So $V - 1$ iterations suffice to set $v.d$ correctly for all v.

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- Then by definition $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$.

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- As BellmanFord does not exit with an error, for all $1 \leq i \leq k$,

$$v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i).$$

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- Summing this inequality over i between 1 and k,

$$\begin{aligned}\sum_{i=1}^{k} v_i.d &\leq \sum_{i=1}^{k} v_{i-1}.d + w(v_{i-1}, v_i) = \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i)\\ &< \sum_{i=1}^{k} v_{i-1}.d = \sum_{i=0}^{k-1} v_i.d.\end{aligned}$$

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- But $v_0 = v_k$, so we have a contradiction. □

Application 1: difference constraints

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- For example, the above system is satisfied by $x_1 = 0$, $x_2 = -1$, $x_3 = 1$, $x_4 = 7$ (among other solutions).
- We will show that this problem can be solved using Bellman-Ford in time $O(nm + n^2)$.

Graph representation of difference constraints

Given m difference constraints in n variables, we create a graph on $n+1$ vertices $v_0, \ldots, v_n$ with $m+n$ edges where:

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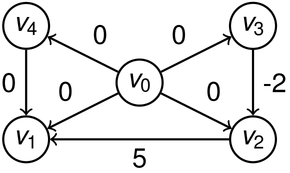
For example:

$$x_1 - x_2 \le 5, \quad x_2 - x_3 \le -2, \quad x_1 - x_4 \le 0$$

corresponds to



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Claim

Let G be the graph corresponding to a system of difference constraints. If G does not contain a negative-weight cycle, the assignment $x_i = \delta(v_0, v_i)$, for all $1 \leq i \leq n$, is a valid solution to the system of constraints.

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- This follows from the triangle inequality

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  and rearranging. □

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- Let $c = v_1, \dots, v_k, v_1$ be an arbitrary cycle on vertices $v_1, \dots, v_k$ (without loss of generality). This corresponds to the inequalities

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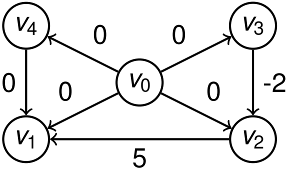
- If there is a valid solution x_i, then all the inequalities are satisfied.
- Summing the inequalities we get 0 for the left-hand side, and the weight of c for the right-hand side.
- So c has weight at least 0, and is not a negative-weight cycle. □

Example

The set of inequalities

$$x_1 - x_2 \leq 5, \quad x_2 - x_3 \leq -2, \quad x_1 - x_4 \leq 0$$

corresponds to the graph



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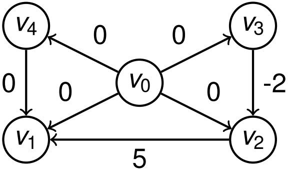

with shortest paths

$$\delta(v_0, v_1) = 0, \quad \delta(v_0, v_2) = -2, \quad \delta(v_0, v_3) = 0, \quad \delta(v_0, v_4) = 0.$$

So

$$x_1 = 0, \quad x_2 = -2, \quad x_3 = 0, \quad x_4 = 0$$

is a solution to the constraints.

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- The running time of Bellman-Ford is thus $O(VE) = O(mn + n^2)$.
- This can be improved to $O(mn)$ time (CLRS exercise 24.4-5).

Application 2: Currency exchange

Imagine we have n different currencies, and a table T whose (i, j)'th entry T_{ij} represents the exchange rate we get when converting currency i to currency j. For example:

	£	$	€
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- We can use Bellman-Ford to determine whether arbitrage is possible.

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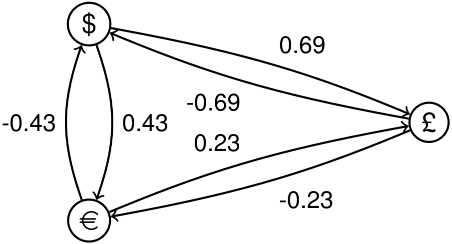
We produce a weighted graph G from the currency table, where the weight of edge $i \to j$ is $-\log_2 T_{ij}$. For example:



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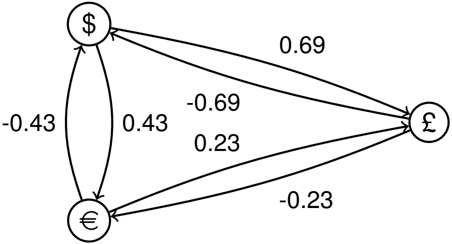

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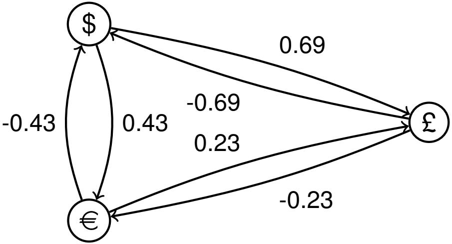

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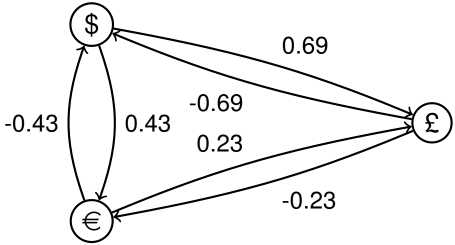

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- This will be negative if and only if $\prod_j T_{c_j c_{j-1}} > 1$, i.e. the sequence of transactions corresponds to an arbitrage opportunity.
- So G has a negative-weight cycle if and only if arbitrage is possible.

Summary

- The Bellman-Ford algorithm solves the single-source shortest paths problem in time $O(VE)$.
- It works if the input graph has negative-weight edges, and can detect negative-weight cycles.
- Although the proof of correctness is a bit technical, the algorithm is easy to implement and doesn't use any complicated data structures.
- It can be used to solve a system of difference constraints and to determine whether arbitrage is possible.

Further Reading

- Introduction to Algorithms
  T.H. Cormen, C.E. Leiserson, R.L. Rivest and C. Stein.
  MIT Press/McGraw-Hill, ISBN: 0-262-03293-7.
  - Chapter 24 – Single-Source Shortest Paths

- Algorithms
  S. Dasgupta, C.H. Papadimitriou and U.V. Vazirani
  `http://www.cse.ucsd.edu/users/dasgupta/mcgrawhill/`
  - Chapter 4, Section 4.6 – Shortest paths in the presence of negative edges

- Algorithms lecture notes, University of Illinois
  Jeff Erickson
  `http://www.cs.uiuc.edu/~jeffe/teaching/algorithms/`
  - Lecture 19 – Single-source shortest paths

Biographical notes

Richard E. Bellman (1920–1984)

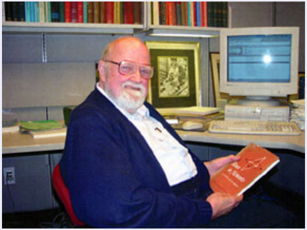
- American mathematician who worked at Princeton, Stanford, the RAND Corporation and the University of Southern California.
- Author of at least 621 papers and 41 books, including 100 papers after the removal of a brain tumour left him severely disabled.
- Winner of the IEEE Medal of Honor in 1979 for his invention of dynamic programming.

Pic: IEEE Global History Network

Biographical notes

Lester Ford, Jr. (1927–)

- Another American mathematician whose other contributions include the Ford-Fulkerson algorithm for maximum flow problems.
- His father was also a mathematician and, at one point, President of the Mathematical Association of America.



Pic: tangrammit.com