The 40^{th} International Workshop on Water Waves and Floating Bodies, 11-14 May 2025, Shanghai, China

Wave propagation through periodic arrays of freely floating rectangular floes

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1 Introduction

Following work presented at the "39th International Workshop on Water Waves and Floating Bodies" on attenuation of waves through continuous broken ice that expanded upon the work of Dafydd & Porter (2024) to consider attenuation in water of finite depth, questions were posed regarding the modelling assumption that the broken ice was continuous (i.e. no gaps) and motion was restricted to heave. In this paper we present an outline of research that has been carried out since to address these questions. We consider two-dimensional wave propagation through broken rectangular floes of ice arranged in a periodic manner in the free surface and free to move in all three degrees of freedom on account of gaps inserted between neighbouring floes. A key aim of the work is to use a small-gap assumption to develop an explicit asymptotic representation for the local dispersion relation which can be used for wave propagation through ice with slowly-varying properties including variable gaps. The mathematical framework exploits the periodicity to study so-called Bloch waves; this is similar to studies such as Linton (2011) and Chou (1998) although the coupling of heave, surge and pitch modes in the study of propagation through periodic arrays appears to be new.

2 Governing equations

Cartesian coordinates (x, z) are chosen with z directed upwards from the rest position of the free surface of an infinitely-deep fluid whose motion is represented by a velocity potential, $\phi(x, z)$, satisfying

$$\nabla^2\phi = 0 \tag{1}$$

in the fluid with $|\nabla\phi| \to 0$ as $z \to \infty$. The periodicity in the array allows us to write $\phi(x+L, z) = \mathrm{e}^{\mathrm{i}kL}\phi(x, z)$ where k is the Bloch wavenumber, to be found. This allows us to consider a single period $0 < x < L$ with

$$\phi(L, z) = \mathrm{e}^{\mathrm{i}kL}\phi(0, z), \quad \phi_x(L, z) = \mathrm{e}^{\mathrm{i}kL}\phi_x(0, z) \tag{2}$$

for $z < -\hat{\rho}d$ and $\hat{\rho} = \rho_i/\rho$ is the ratio of ice and water densities. The combined linearised kinematic and dynamic free surface conditions can be expressed as

$$\phi_z - K\phi = 0, \quad \text{on} \quad z = 0,\ 0 < x < \ell \tag{3}$$

where $K = \omega^2/g$. Kinematic conditions apply on the walls and base of the floe and dynamic conditions apply to the heave, surge and pitch (h, s, p hereafter) motions in terms of their respective amplitudes ζ, ξ, $a\theta$ where $a = L - \ell$. The decomposition

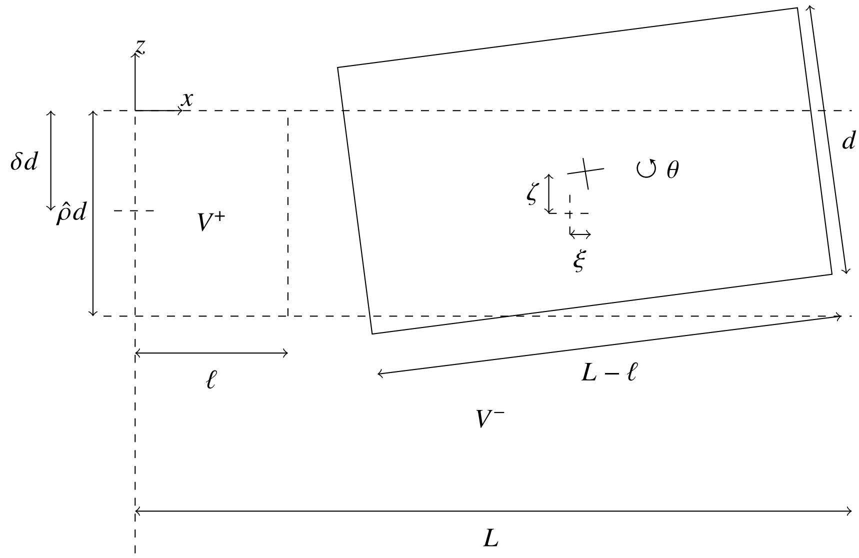
$$\phi(x, z) = -\mathrm{i}\omega\{\zeta\phi^{(h)}(x, z) + \xi\phi^{(s)}(x, z) + a\theta\phi^{(p)}(x, z)\} \tag{4}$$

ascribes the three potentials $\phi^{(h,s,p)}$ to forced motions of unit amplitude satisfying

$$\phi_z^{(h)}(x, -\hat{\rho}d) = 1, \quad \phi_z^{(s)}(x, -\hat{\rho}d) = 0, \quad \phi_z^{(p)}(x, -\hat{\rho}d) = -(x - x^c)/a \tag{5}$$

for $\ell < x < L$ and, for $-\hat{\rho}d < z < 0$,

$$\phi_x^{(h)}(\ell, z) = 0, \quad \phi_x^{(s)}(\ell, z) = 1, \quad \phi_x^{(p)}(\ell, z) = (z - z^c)/a \tag{6}$$



where $x^c = \frac{1}{2}(L+\ell)$, $z^c = (\frac{1}{2}-\hat\rho)d$ in addition to the previous requirements placed upon ϕ. The dynamic conditions are represented by the homogeneous system of equations

$$\begin{pmatrix} 1-Kd(\hat\rho+F^{(h,h)}) & -KdF^{(s,h)} & -KdF^{(p,h)} \\ -KdF^{(h,s)} & -Kd(\hat\rho+F^{(s,s)}) & -KdF^{(p,s)} \\ -KdF^{(h,p)} & -KdF^{(s,p)} & \frac{1}{12}-\frac{1}{2}\hat\rho(1-\hat\rho)\hat{d}^2 \\ & & -Kd(\frac{1}{12}\hat\rho(1+\hat{d}^2)+F^{(p,p)}) \end{pmatrix} \begin{pmatrix} \zeta \\ \xi \\ a\theta \end{pmatrix} = 0 \tag{7}$$

where $\hat{d} = d/a$ is the aspect ratio of the flow and where we have defined the normalised dynamic force in mode $b \in \{h,s,p\}$ due to forced motion in mode $a \in \{h,s,p\}$ by

$$\begin{aligned} F^{(a,b)} &= \frac{1}{da}\int_\ell^L \phi^{(a)}(x,-\hat\rho d)\overline{\phi_z^{(b)}}(x,-\hat\rho d)\,\mathrm{d}x \\ &+ \frac{1}{da}\int_{-\hat\rho d}^0 \phi^{(a)}(\ell,z)\overline{\phi_x^{(b)}}(\ell,z) - \phi^{(a)}(0,z)\overline{\phi_x^{(b)}}(0,z)\,\mathrm{d}z, \end{aligned} \tag{8}$$

the overline representing complex conjugation. For a given frequency K, the non-trivial solutions of (7) satisfied by real k represent propagating wave solutions.

3 Solutions in heave, surge and pitch

In the fluid region $V^+ := \{0<x<\ell, -\hat\rho d<z<0\}$ a separation series expansion of $\phi^{(h,s,p)}$ satisfying both (1), (3), (2) and (6) is given by

$$\phi^{(h,s,p)}(x,z) = \psi^{(h,s,p)}(x,z) + \sum_{m=0}^{\infty} a_m Z_m(z)\cos\alpha_m x \tag{9}$$

where $\alpha_m = m\pi/\ell$. In $V^- = \{0<x<L, z<-\hat\rho d\}$, the general solution is required to satisfy (1), (2) and (5) and thus write

$$\phi^{(h,s,p)}(x,z) = \varphi^{(h,s,p)}(x,z) + \sum_{m=-\infty}^{\infty} b_m \mathrm{e}^{\mathrm{i}\beta_m x}\mathrm{e}^{\beta_m(z+\hat\rho d)} \tag{10}$$

where $\beta_m = k + 2\pi m/L$. The functions $\psi^{(h,s,p)}(x,z)$ and $\varphi^{(h,s,p)}(x,z)$ are particular solutions defined in the regions V^+ and V^- that account for the inhomogeneous boundary conditions

(5) and (6) as well as satisfying (1), (3) and (2) and decaying with depth. For each particular solution we find it advantageous to prescribe homogeneous Neumann conditions on $z = -\hat{\rho}d$, $0 < x < \ell$. If there is no inhomogeneity in either V^- or V^+ the corresponding particular solution is simply zero.

Applying continuity of $\phi^{(h,s,p)}$ and $\phi_z^{(h,s,p)}(x, -\hat{\rho}d) \equiv W^{(h,s,p)}(x)$ across the interface $z = -\hat{\rho}d$, $0 < x < \ell$ reduces each problem to an integral equation

$$\frac{1}{\ell}\int_0^\ell W^{(h,s,p)}(x')T(x,x')\,\mathrm{d}x' = G^{(h,s,p)}(x), \quad 0 < x < \ell, \tag{11}$$

where $G^{(h,s,p)}(x)$ are known functions that relate to the particular solutions and

$$T(x,x') = \frac{K\hat{\rho}d - 1}{K\ell} + 2\sum_{m=1}^{\infty}\frac{(K\tanh(\alpha_m\hat{\rho}d) - \alpha_m)\cos\alpha_m x\cos\alpha_m x'}{\alpha_m\ell\,(K - \alpha_m\tanh(\alpha_m\hat{\rho}d))} + \sum_{m=-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\beta_m(x-x')}}{\beta_m L}$$

is the integral kernel, common to all problems. The quantities of interest needed to compute the determinant of the matrix (8) are found to be

$$F^{(a,b)} = -\frac{\ell}{ad}\int_0^\ell W^{(a)}(x)\overline{G^{(b)}(x)}\,\mathrm{d}x + \text{terms originating from particular solutions.} \tag{12}$$

4 A small gap approximation

Assuming the gaps are small, i.e. $\ell/d = \epsilon \ll 1$ whilst avoiding resonance in the fluid channels ($|K\hat{\rho}d - 1| = O(1)$) we can approximate

$$T(x,x') \approx \frac{1}{\epsilon}\widehat{T}, \qquad \text{where } \widehat{T} = \frac{K\hat{\rho}d - 1}{Kd} \tag{13}$$

whilst it can be shown that $G^{(h)}(x) \approx \widehat{G}^{(h)}/\epsilon$, $G^{(s,p)}(x) \approx \widehat{G}^{(s,p)}/\epsilon^2$ where $\widehat{G}^{(h,s,p)}$ can all be found explicitly. Using these in (11) and (12) allows us to make small-gap approximations to $F^{(a,b)}$. For example, we find

$$F^{(h,h)} \approx \frac{1}{2}(L/d - \epsilon)\frac{\cot(kL/2)}{2} + \epsilon\frac{KL}{4(1 - K\hat{\rho}d)\sin^2(kL/2)} \tag{14}$$

to $O(\epsilon)$, with more complicated expressions for the remaining force coefficients. For constrained heave motions ξ, $\theta = 0$ and (7) reduces to $Kd(\hat{\rho} + F^{(h,h)}) = 1$ which gives

$$kL \approx 2\tan^{-1}(\tilde{K}) + \epsilon\frac{2d\tilde{K}^3}{L(1 + \tilde{K}^2)} \tag{15}$$

where $\tilde{K} = KL/2(1 - K\hat{\rho}d)$ and which tends to the well-known mass-loading result $k = K/(1 - K\hat{\rho}d)$ in the zero-gap, low frequency limit.

5 Numerical results

We compute accurate values of $F^{(a,b)}$ in (12) by solving (7) using a Galerkin method in which we write

$$W^{(h,s,p)}(x) \approx \sum_{n=0}^{N} {A_n}^{(h,s,p)} u_n(x) \tag{16}$$

where $A_n^{(h,s,p)}$ are expansion coefficients and $u_n(x)$ are functions which incorporate the anticipated inverse cube root singularity at the corners of the rectangular floes.

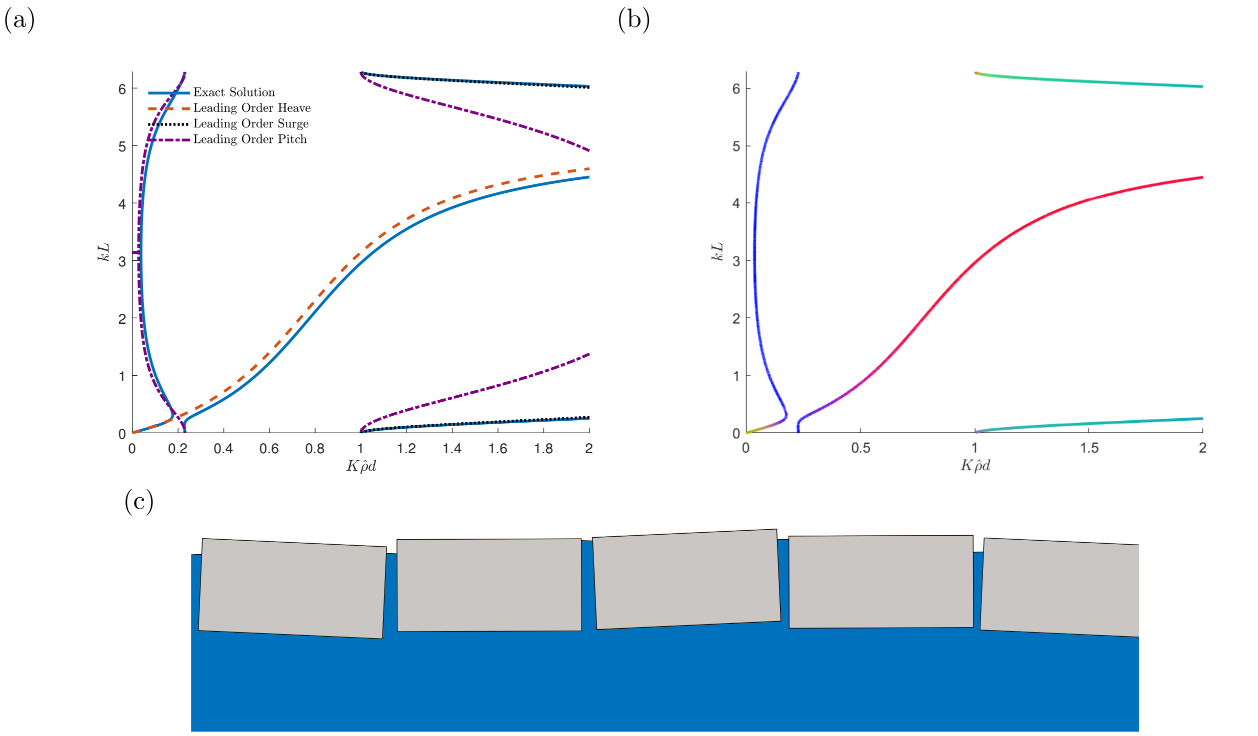


Figure 1: (a) Asymptotic model compared against exact Galerkin computations, $\epsilon = 0.01$, (b) Colour-coded plots for the heave-surge-pitch model where Red: Heave, Green: Surge, Blue: Pitch, $\epsilon = 0.01$, (c) Snapshot of moment in time with many periodic floes moving with a frequency corresponding to heave, surge and pitch modes, $\epsilon = 0.12$.

In Figure 1(a) we compare numerical results with the explicit small-gap asymptotic results in the case of $\epsilon = 0.01$ and good agreement is found. The results show the variation of Bloch wavenumber $kL \in (0, 2\pi)$ against frequency parameter $K\hat{\rho}d$. Figure 1(b) uses RGB colouring to illustrate which of the heave/surge/pitch modes are dominant and, interestingly, we see that most regimes have multiple possible dominant modes, with heave and pitch being present largely throughout and surge dominant modes only occurring as the frequency increases and passes through resonance, $K\hat{\rho}d = 1$,. The small gaps make minor contributions to the dispersion relations for heave-dominant modes in the long-wavelength regime, but pitch-dominant modes are also present there and cannot be discarded. Further results will be shown at the Workshop including further comparisons to the heave-restricted continuous model and demonstrations on how fully animated movement profiles can be extracted from this data.

References

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