Wave energy extraction by a parallel array of bottom-hinged plates

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Highlights

In a recent study by authors [1], wave scattering by a discrete plate-array metacylinder of arbitrary cross-section was studied. A Fourier transform-based method has been developed to reduce the plate influence in 2D to a 1D wave equation solved via a Green's function. Building on this framework, we extend the method to study wave radiation by an array of bottom-hinged plates, via incorporating evanescent modes, which is essential for the study of energy extraction. Particular focus is placed to the hydrodynamic coefficients by the hinged plates.

1 Statement of the problem

The fluid, with a constant depth h, is assumed to be incompressible and inviscid, and the flow is considered irrotational, allowing for the existence of a velocity potential Φ. A 3D coordinate system $Oxyz$ is defined with the Oxy plane coinciding with the undisturbed free surface and Oz axis pointing upward. We consider a train of water waves interacting with a parallel array of $N+1$ thin vertical plates, each hinged at the bottom. The plates are located at $x = x_j$, extend horizontally within $y \in (-b_j, b_j)$ and stand vertically $-h < z < 0$ with $j = 0, 1, ..., N$. The velocity potential in a steady state of an angular frequency ω is written as $\Phi(x, y, z, t) = \mathrm{Re}[\phi(x, y, z)\, \mathrm{e}^{-\mathrm{i}\omega t}]$, where $\phi(x, y, z)$ can be decomposed into

$$\phi(x,y,z) = \frac{-\mathrm{i}gA}{\omega\psi_0(0)}\psi_0(z)\left[\varphi_{\mathrm{inc}}(x,y) + \sum_{n=0}^{N}\varphi_{\mathrm{sca}}^{(n)}(x,y)\right] + \sum_{n=0}^{N}\phi_{\mathrm{rad}}^{(n)}(x,y,z). \tag{1}$$

Here, A is the wave amplitude, and $\varphi_{\mathrm{inc}}(x, y) = \mathrm{e}^{\mathrm{i}k(x\cos\theta_0 + y\sin\theta_0)}$ is the incident wave potential, where k denotes the wavenumber satisfying the dispersion relation $\omega^2 = gk\tanh(kh)$, and θ_0 is the incident angle with respect to positive x-axis. Besides, $\psi_j(z)$ represent the vertical mode functions

$$\psi_j(z) = N_j^{-1/2}\cos[k_j(z+h)] \ \text{ with } \ N_j = \frac{1}{2} + \frac{\sin(2k_jh)}{4k_jh} \ \text{ satisfying } \ \frac{1}{h}\int_{-h}^{0}\psi_i(z)\psi_j(z)\mathrm{d}z = \delta_{i,j}, \tag{2}$$

where k_j are positive roots of $\omega^2 = gk_j\tan(k_jh)$ for $j > 1$, and $k_0 = -\mathrm{i}k$ is defined. The wave scattering problem has been studied in [1], and thus we only focus on the wave radiation problem here. The radiation potential can be expanded as

$$\phi_{\mathrm{rad}}^{(n)}(x,y,z) \approx \sum_{j=0}^{J}\Omega_n V_j\, \varphi_j^{(n)}(x,y)\, \psi_j(z) \ \text{ so that } \ \partial_x\varphi_j^{(n)}(x_m,y) = \delta_{n,m} \ \text{ for } \ y \in (-b_m, b_m), \tag{3}$$

where Ω_n denotes the angular velocity of pitch motion by the plate n, and $u(z)$ represents the velocity induced by the paddle's pitch motion at a unit angular velocity, expanded as

$$u(z) = z + h \approx \sum_{j=0}^{J} V_j \, \psi_j(z) \quad \text{with} \quad V_j = \frac{k_j h \sin(k_j h) + \cos(k_j h) - 1}{k_j^2 h} N_j^{-1/2}. \tag{4}$$

Then, the radiation potential is governed by a 2D Helmholtz equation over the horizontal plane

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - k_j^2 \right) \varphi_j^{(n)} = 0. \tag{5}$$

2 Wave radiation by an array of plates in open water

We now consider the radiation of waves by an array of bottom-hinged plates in open water. Following [1], we perform the Fourier transform along the plate

$$\bar{\varphi}_j^{(n)}(x;\beta) = \int_{-\infty}^{\infty} \varphi_j^{(n)}(x,y) \mathrm{e}^{-\mathrm{i}\beta y} \mathrm{d}y \;\; \text{and} \;\; \varphi_j^{(n)}(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\varphi}_j^{(n)}(x;\beta) \mathrm{e}^{\mathrm{i}\beta y} \mathrm{d}\beta \tag{6}$$

Then the governing wave equation is reduced to a 1D Helmholtz equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma_j^2 \right) \bar{\varphi}_j^{(n)} = 0, \;\; \text{where} \;\; \gamma_0 = \begin{cases} \sqrt{\beta^2 - k^2} & |\beta| > k \\ -\mathrm{i}\sqrt{k^2 - \beta^2} & \text{otherwise} \end{cases} \;\; \text{and} \;\; \gamma_j = \sqrt{k_j^2 + \beta^2}. \tag{7}$$

The transformation of the plate conditions leads to

$$\frac{\mathrm{d}}{\mathrm{d}x} [\bar{\varphi}_j^{(n)}(x_n^+;\beta) - \bar{\varphi}_j^{(n)}(x_n^-;\beta)] = 0 \;\; \text{and} \;\; \bar{\varphi}_j^{(n)}(x_n^+;\beta) - \bar{\varphi}_j^{(n)}(x_n^-;\beta) = P_j^{(n)}(\beta) \;\; \text{for} \;\; y \in (-b_n, b_n), \tag{8}$$

where $P_j^{(n)}(\beta)$ is the Fourier transform of potential jump across the plate, defined as

$$P_j^{(n)}(\beta) = \int_{-b_n}^{b_n} p_j^{(n)}(y) \mathrm{e}^{-\mathrm{i}\beta y} \mathrm{d}y \;\; \text{with} \;\; \phi(x_n^+, y) - \phi(x_n^-, y) = \begin{cases} p_j^{(n)}(y) & |y| < b_n \\ 0 & |y| > b_n \end{cases} \tag{9}$$

By using the 'dipole-like' fundamental solution to the 1D Helmholtz equation satisfying the continuous $\mathrm{d}_x[g_j(x_n^+, x_n;\beta) - g_j(x_n^-, x_n;\beta)] = 0$ and potential jump $g_j(x_n^+, x_n;\beta) - g_j(x_n^-, x_n;\beta) = -1$ conditions, given by

$$g_j(x, x_n;\beta) = -\frac{1}{2} \operatorname{sgn}(x - x_n) \, \mathrm{e}^{-\gamma_j |x - x_n|}, \tag{10}$$

we construct the solution $\bar{\varphi}_j^{(n)}(x,\beta)$ by a superposition of Green's function

$$\bar{\varphi}_j^{(n)}(x,\beta) = \sum_{n=0}^{N} P_j^{(n)}(\beta) g_j(x, x_n;\beta) = -\frac{1}{2} \sum_{n=0}^{N} P_j^{(n)}(\beta) \operatorname{sgn}(x - x_n) \, \mathrm{e}^{-\gamma_j |x - x_n|}. \tag{11}$$

By imposing the boundary condition (3) on the hinged paddle, we have the integral equations

$$\frac{1}{4\pi} \sum_{n=0}^{N} \int_{-\infty}^{\infty} \int_{-b_n}^{b_n} \gamma_j \, p_j^{(n)}(y') \mathrm{e}^{-\gamma_j |x_m - x_n| + \mathrm{i}\beta(y - y')} \mathrm{d}y' \mathrm{d}\beta = \delta_{n,m}. \tag{12}$$

The solution is expanded as

$$p_j^{(n)}(y) \approx \sum_{q=0}^{Q} a_q^{(j,n)} w_q(y/b_n) \;\text{ with }\; w_q(u) = \frac{\mathrm{e}^{\mathrm{i}\frac{q\pi}{2}}}{(q+1)\pi} \sqrt{1-u^2}\, \mathrm{U}_q(u), \tag{13}$$

where $\mathrm{U}_q(u)$ denotes the Chebyshev polynomial of the second kind. Substituting the expansion (13) into the boundary integral equation (12) and implementing Galerkin's method via integrating the test function $w_p^*(y/b_m)$ over $y \in [-b_m, b_m]$, we have the following equation system

$$\sum_{n=0}^{N} \sum_{q=0}^{Q} a_q^{(j,n)} K_{q,n;p,m}^{(j)} = \delta_{n,m}\delta_{p,0}\frac{b_m}{2} \;\text{ with }\; K_{q,n;p,m}^{(j)} = \frac{b_m b_n}{4\pi} \int_{-\infty}^{\infty} \gamma_j D_p(\beta b_m) D_q(\beta b_n) \mathrm{e}^{-\gamma_j |x_m - x_n|} \mathrm{d}\beta, \tag{14}$$

where function $D_q(\beta b_n)$ is defined as

$$D_q(\beta b_n) = \frac{1}{b_n} \int_{-b_n}^{b_n} \mathrm{e}^{-\mathrm{i}\beta\eta} w_q(\eta/b_n) \mathrm{d}\eta = \begin{cases} \mathrm{J}_{q+1}(\beta b_n)/(\beta b_n) & \beta \neq 0, \\ \delta_{q,0}/2 & \beta = 0. \end{cases} \tag{15}$$

The integral equation (14) is consistent with [2], which is based on recursive matching of expansions across different domains. Notably, the equations for vertical modes j are uncoupled, indicating the computational cost is linearly dependent on the number of vertical modes. Algorithms for efficiently computing the wavenumber integral can be found in [1] by using the orthogonal relation of Bessel function and encoding the symmetric and asymmetric components. For the present radiation problem, only the symmetric components are relevant.

3 An infinite periodic array

The method is straightforwardly generalised to an infinite periodic array of plates extending repeatedly in the y-direction with a spacing of $2d$. Due to the periodicity, this problem is equivalent to wave radiation in a uniform channel of width $2d$. This problem was considered in [3], which used an infinite series of image sources to represent channel wall effects. However, it is a nontrivial task to handle the infinite sum of Hankel functions due to the slow convergence of the series.

Such difficulty can be effectively circumvented in the present study. Unlike the continuous Fourier transform used in open waters, the present problem involves applying a Fourier transform over a finite interval

$$\bar{\varphi}_{j,l}^{(n)}(x;\beta) = \frac{1}{2d} \int_{-d}^{d} \varphi_j^{(n)}(x,y) \mathrm{e}^{-\mathrm{i}\beta_l y} \mathrm{d}y \;\text{ and }\; \varphi_j^{(n)}(x,y) = \sum_{l=-\infty}^{\infty} \bar{\varphi}_{j,l}^{(n)}(x;\beta) \mathrm{e}^{\mathrm{i}\beta_l y}, \tag{16}$$

where $\beta_l = l\pi/d$, and then the governing equation becomes a 1D Helmholtz equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma_{j,l}^2\right) \bar{\varphi}_{j,l}^{(n)} = 0, \;\text{ where }\; \gamma_{0,l} = \begin{cases} \sqrt{\beta_l^2 - k^2} & |\beta| > k \\ -\mathrm{i}\sqrt{k^2 - \beta_l^2} & \text{otherwise} \end{cases} \;\text{ and }\; \gamma_{j,l} = \sqrt{k_j^2 + \beta_l^2}. \tag{17}$$

In analogy to Eq. (11), the solution can be constructed in the form

$$\bar{\varphi}_{j,l}^{(n)}(x,\beta) = -\frac{1}{2} \sum_{n=0}^{N} P_j^{(n)}(\beta_l) \operatorname{sgn}(x - x_n)\, \mathrm{e}^{-\gamma_{j,l}|x - x_n|} \;\text{ with }\; P_j^{(n)}(\beta_l) = \frac{1}{2d} \int_{-b_n}^{b_n} p_j^{(n)}(y) \mathrm{e}^{-\mathrm{i}\beta_l y} \mathrm{d}y. \tag{18}$$

By using the solution expansion in Eq. (13) and applying the bottom-hinged plate boundary condition (3) in the sense of the Galerkin method, we obtain a system of equations analogous to Eq. (14), with the coefficients replaced by a series as follows

$$K^{(j)}_{q,n;p,m} = \frac{b_m b_n}{4d} \sum_{l=-\infty}^{\infty} \gamma_{j,l} D_p(\beta_l b_m) D_q(\beta_l b_n) \mathrm{e}^{-\gamma_{j,l}|x_m - x_n|}. \tag{19}$$

4 Hydrodynamic coefficients

Once the unknown coefficients $a_q^{(j,n)}$ are solved, we can obtain the wave exciting pitch moment and the added mass and wave-radiation damping, expressed as

$$M_x^{(n)} = \frac{-\rho g A b_n h}{\cosh(kh)} a_0^{(\mathrm{s,n})} N_0^{1/2} V_0/2 \quad \text{and} \quad \mathrm{i}\omega A_{mn} - B_{mn} = -\mathrm{i}\rho\omega b_m h \sum_{j=0}^{J} V_j^2\, a_0^{(j,n)}/2. \tag{20}$$

With the hydrodynamic coefficients associated with radiation and diffraction, the wave-induced motions can be calculated, enabling the quantification of energy harvesting. This work is still ongoing, and we present only preliminary results. Figure 1 depicts the wave exiting pitch moment (left) and added mass and radiation damping (right) by a single bottom-hinged plate in both open water and channel configurations. Vertical lines indicate the wavenumbers corresponding to the tank crossing modes, $k = \pi/d$. The added mass (solid line) and damping (dashed line) are nondimenisonalised as $\tilde{A} = A/(4\rho b^2 h^3)$ and $\tilde{B} = B/(4\omega\rho b^2 h^3)$, respectively. Sharp variations in the hydrodynamic coefficients are observed near the first crossing mode. More results including the wave energy harnessing will be presented at the workshop.

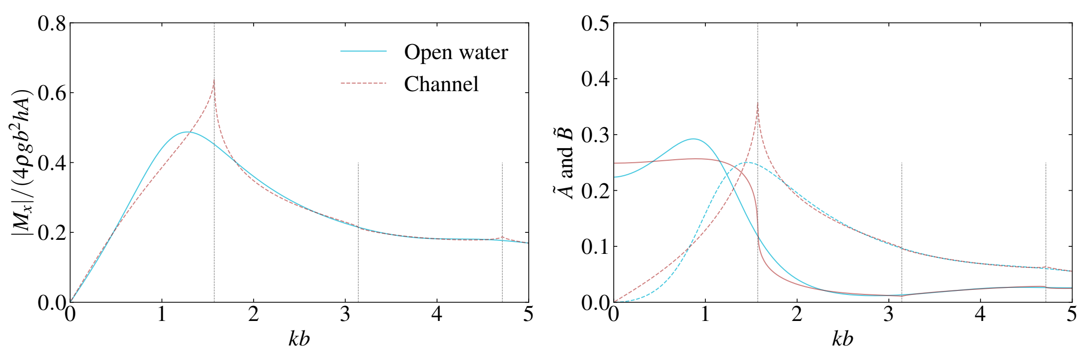

Figure 1: Nondimensional pitch moment (left) and added mass and radiation damping (right) by a bottom-hinged plate for $b/h = 1.0$ and $b/d = 0.5$ in both open water and channel scenarios.

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REFERENCES

[1] Liang, H., Porter, R., and Zheng, S. 2024. *Wave scattering by plate array metacylinders of arbitrary cross-section*. *Journal of Fluid Mechanics 1001*, A6.

[2] Noad, I. F., and Porter, R. 2015. *Optimisation of arrays of flap-type oscillating wave surge converters*. *Applied Ocean Research 50*, 237–253.

[3] Renzi, E., and Dias, F. 2012. *Resonant behaviour of an oscillating wave energy converter in a channel*. *Journal of Fluid Mechanics 701*, 482–510.