Extending limits on wave power absorption by axisymmetric devices

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1 Introduction

The theoretical limits for absorption of energy in monochomatic water waves of wavelength λ by axisymmetric wave energy converters (WECs) operating in rigid-body motion were established independently in the 1970s by a number of authors ([1], [2], [3]). The maximum mean power generated by an axisymmetric WEC device absorbing due to heave motion is equivalent to that contained in $\lambda/2\pi$ length of incident wave crest. For devices absorbing through surge and/or pitch motions the maximum so-called capture width doubles to λ/π. For devices that absorb in both heave and surge/pitch the maximum capture width increases further to $3\lambda/2\pi$.

The purpose of the present paper is to demonstrate that these rigid-body limits can be extended without bound (theoretically at least) by allowing an axisymmetric device to operate and absorb energy in "generalised hydrodynamic modes" of motion (following ideas promoted by [4], although the general idea had previously been applied to wave power absorption by hinged rafts by Newman and Mei; see [5]). This involves allowing the surface of the device to move with more than the translational and rotational degrees of freedom that would be in operation if the device were rigid.

In this paper the general theory is applied to a specific example in which an array of narrow vertical paddles are distributed evenly around the surface of a vertical cylinder which itself extends through the fluid depth. The paddles are fitted with springs and dampers extract power from the waves. The simplicity of the geometry combined with a continuum approximation for the narrow paddles is used to devise strategies for assigning spring and damper characteristics to optimise the power.

2 General theory

We operate under classical linearised water wave theory in which the flow can be described by a velocity potential satisfying

$$\nabla^2\phi = 0, \qquad \text{in the fluid} \tag{1}$$

with

$$\phi_z = 0, \qquad \text{on } z=-h \qquad \text{and} \qquad \phi_z - (\omega^2/g)\phi = 0, \qquad \text{on } z=0 \tag{2}$$

assuming the factorisation of a time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$ where ω is the angular frequency of motion.

A plane wave (pw) of wavelength $\lambda = 2\pi/k$, angular frequency ω and amplitude A travelling in the positive x-direction on water of density ρ and depth h is described by the velocity potential

$$\phi_{pw}(x,y,z) = -\frac{\mathrm{i}Ag}{\omega}\mathrm{e}^{\mathrm{i}kx}\psi_0(z) \tag{3}$$

where $\omega = \sqrt{gk\tanh kh}$ is related to the wavenumber k and $\psi_0(z) = \cosh k(z+h)/\cosh kh$ is the eigenfunction through the depth, z, associated with propagating waves. The power per unit crest length of wave is

$$\mathbb{P}_{pw} = \frac{1}{2}\rho g|A|^2 c_g, \qquad \text{where} \qquad c_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{1}{2}(\omega/k)(1+2kh/\sinh 2kh). \tag{4}$$

The plane wave in (3) can be written as a sum of incoming and outgoing circular waves

$$\phi_{pw}(r,\theta,z) = \phi_{in}(r,\theta,z) + \phi_{out}(r,\theta,z) \tag{5}$$

where

$$\phi_{in} = -\frac{\mathrm{i}Ag}{2\omega}\psi_0(z)\sum_{n=0}^{\infty}\epsilon_n \mathrm{i}^n H_n^{(2)}(kr)\cos n\theta \qquad \text{and} \qquad \phi_{out} = -\frac{\mathrm{i}Ag}{2\omega}\psi_0(z)\sum_{n=0}^{\infty}\epsilon_n \mathrm{i}^n H_n^{(1)}(kr)\cos n\theta \tag{6}$$

in terms of Hankel functions and where $\epsilon_0 = 1$, $\epsilon_n = 2$ for $n \geq 1$. It is readily established that the flux of energy in the nth circular component of each of these components is $P_n = (\epsilon_n \lambda/2\pi)\mathbb{P}_{pw}$. When the incident plane wave interacts with a structure be it fixed, freely floating, under mooring constraints, or absorbing energy, the total potential in the far field can be expressed as

$$\phi(r,\theta,z) \sim \phi_{pw} - \frac{\mathrm{i}Ag}{\omega}\psi_0(z)\sum_{n=0}^{\infty}\epsilon_n \mathrm{i}^n a_{n,0} H_n^{(1)}(kr)\cos n\theta = \phi_{in} - \frac{\mathrm{i}Ag}{2\omega}\psi_0(z)\sum_{n=0}^{\infty}\epsilon_n \mathrm{i}^n (2a_{n,0}+1) H_n^{(1)}(kr)\cos n\theta \tag{7}$$

and thus the power lost to the structure is

$$P = \sum_{n=0}^{\infty} P_n \left(1 - |2a_{n,0}+1|^2\right). \tag{8}$$

It follows that the scattering coefficients $a_{n,0} = a_{n,0}^S$ (say) for non-absorbing structures must satisfy the condition $|2a_{n,0}^S + 1| = 1$. However, for devices with the capacity to absorb energy, it is possible to absorb all of the available power, P_n, from the nth mode if the scattering coefficients can be made to satisfy $2a_{n,0} + 1 = 0$. For this is to happen the device must have the capacity to radiate waves through motions responsible for absorbing wave energy in the nth circular mode, i.e. in proportion to $\cos n\theta$. For example, rigid-body heave motion radiates waves only in the zeroth circular mode, and so its maximum power absorption is limited to $P_{max} = P_0$, whilst surge and pitch motions radiate in the $n = 1$ circular mode giving rise to a maximum of $P_{max} = P_1$; combined heave and surge/pitch provide a maximum of $P_{max} = P_0 + P_1$. Thus we recover the classical results stated in the opening paragraph of the abstract.

However, if we are able to design a WEC whose absorbing components radiate in all $0 \leq n \leq M$ circular modes, then there is the possibility of tuning the device operation to give

$$P_{max} = \frac{\mathbb{P}_{pw}\lambda}{\pi}(M + \tfrac{1}{2}). \tag{9}$$

3 A cylindrical wave energy converter

A vertical cylinder of radius a extends through depth. An array of N identical narrow vertical paddles are attached to the surface of the cylinder having width $2\pi a/N \ll c$, their length. The nth paddle is centred on the axial planes $\theta_n = (2n-1)\pi/N$, $n = 1, 2, \ldots, N$ and, in motion, oscillates radially either in piston-like motion, or through a pitch rotation about a hinged lower edge, with amplitude $\mathrm{Re}\{\sigma_n \mathrm{e}^{-\mathrm{i}\omega t}\}$. The nth paddle is connected to the cylinder with a spring having spring constant κ_n and a linear damper with damping rate γ_n through which power is extracted.

The kinematic boundary condition on the nth paddle is therefore

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=a} = -\mathrm{i}\omega\sigma_n f(z)\cos(\theta - \theta_n), \qquad -h < z < 0, \quad \theta_n - \pi/N < \theta < \theta_n + \pi/N \tag{10}$$

for $n = 1, 2, \ldots, N$ and $\cos(\theta - \theta_n)$ is a geometric factor due to the assumed curved surface of the paddle whilst $f(z)$ encodes the paddle's vertical displacement. For e.g. piston-like operation is given by $f(z) = 1$, $-c < z < 0$ and $f(z) = 0$, $-h < z < -c$. The equation of motion for the nth paddle is expressed by

$$-\omega^2(2\pi a\mathcal{M}/N)\sigma_n = \mathrm{i}\omega\gamma_n\sigma_n - (\kappa_n + (2\pi a\mathcal{C}/N))\sigma_n - \mathrm{i}\omega\rho\int_{-h}^{0}\int_{\theta_n-\pi/N}^{\theta_n+\pi/N} \phi(a,\theta,z) f(z)\cos(\theta-\theta_n)\, a d\theta\, dz \tag{11}$$

where $\mathcal{M}$ is its mass (or moment of inertia) and $\mathcal{C}$ is its buoyancy force (or moment) per unit width.

Since N is large and the paddles are narrow we assume that all discrete variables, σ_n, κ_n and γ_n may be replaced by discrete evaluations, $\sigma(\theta_n)$, $(2\pi a/N)\kappa(\theta_n)$ and $(2\pi a/N)\gamma(\theta_n)$ of (scaled) continuous functions allowing (10) and (11) to be approximated and combined into a single coupled dynamic and kinematic boundary condition on $r = a$ for $-h < z < 0$ and $0 \leq \theta < 2\pi$, namely

$$[\mathcal{M} - \omega^{-2}(\kappa(\theta) + \mathcal{C}) + \mathrm{i}\omega^{-1}\gamma(\theta)] \left.\frac{\partial \phi}{\partial r}\right|_{r=a} = \rho f(z) \int_{-h}^{0} \phi(a, \theta, z) f(z)\, dz. \tag{12}$$

The general solution to (1), (2) with (12) subject to an incident plane wave can be expressed as

$$\phi(r, \theta, z) = -\frac{\mathrm{i}gA}{\omega} \sum_{n=0}^{\infty} \epsilon_n \mathrm{i}^n \phi_n(r, z) \cos n\theta \tag{13}$$

where

$$\phi_n(r, z) = (J_n(kr) + a_{n,0} H_n^{(1)}(kr))\psi_0(z) + (J_n'(ka) + a_{n,0} H_n^{(1)'}(ka)) \sum_{m=1}^{\infty} \frac{kF_m N_0 K_n(k_m r)}{k_m F_0 N_m K_n'(k_m a)} \psi_m(z) \tag{14}$$

in terms of Bessel and modified Bessel functions and

$$F_n = \frac{1}{h} \int_{-h}^{0} \psi_n(z) f(z)\, dz, \qquad \text{and} \qquad N_n = \frac{1}{h} \int_{-h}^{0} [\psi_n(z)]^2\, dz \tag{15}$$

for $n = 0, 1, 2, \ldots$ and, for $n \geq 1$, $\psi_n(z) = \cos k_n(z+h)/\cos k_n h$ where $\mathrm{i}k_n$ (k_n real and positive) are roots of the dispersion relation. The scattering coefficients, $a_{n,0}$, are determined from the application of (12) to (13) and provides the platform for devising strategies to optimise power absorption by the paddles.

We consider two strategies. The first assumes all springs and dampers are identical. Then $\kappa(\theta) \equiv \kappa$, $\gamma(\theta) \equiv \gamma$ in (12) and by imposing $a_{m,0} = -\frac{1}{2}$ for an m of our choice, we can guarantee that 100% of the power, P_m, is absorbed from the mth circular mode such that $P \geq P_m$. In doing so, we find

$$\gamma = \frac{2\omega\rho h F_0^2}{\pi k^2 a N_0 |H_m^{(2)'}(ka)|^2} \tag{16}$$

whilst κ is determined from

$$\frac{\mathcal{M} - \omega^{-2}(\kappa + \mathcal{C})}{\rho h} = \frac{F_0^2(J_m(ka)J_m'(ka) + Y_m(ka)Y_m'(ka))}{kN_0|H_m^{(2)'}(ka)|^2} + \sum_{n=1}^{\infty} \frac{F_n^2 K_m(k_n a)}{k_n N_n K_m'(k_n a)}. \tag{17}$$

In the second strategy, we allow the springs and dampers to vary around the cylinder, expressing them both as Fourier series. It can subsequently be shown that it is possible to determine their Fourier coefficients and hence the design of springs and dampers such that $P = P_{max}$ defined by (9) by setting $a_{n,0} = -\frac{1}{2}$ for $n \leq M$ and $a_{n,0} = 0$ for $n > M$. In principle, there is no bound upon M although there will be practical considerations associated with increasing values of M.

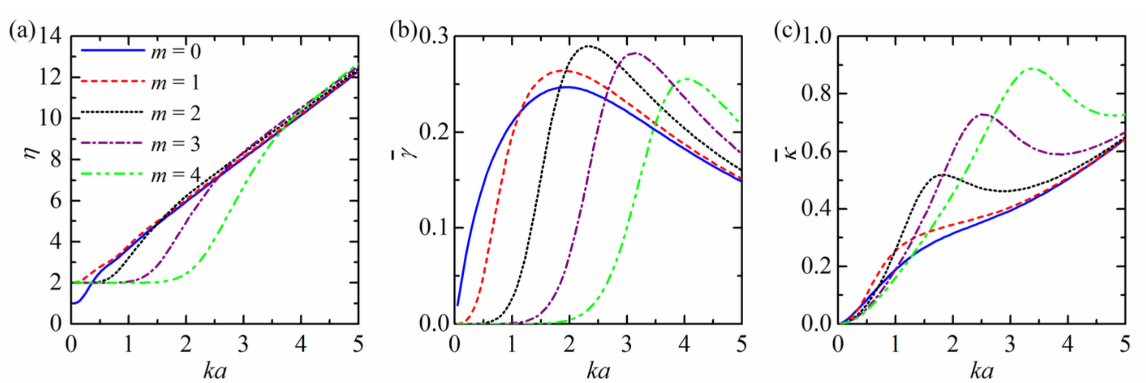

Figure 1: (a) Capture factor, $\eta = P/P_0 = P/(\mathbb{P}_{pw}\lambda/2\pi)$, against ka for piston-like paddle motion and, in (b), (c) corresponding dimensionless optimised damper and spring values $\bar{\kappa} = \kappa/(\rho g a)$, $\bar{\gamma} = \gamma/(\rho a\sqrt{gh})$.

4 Results

In Fig. 1 we show the dimensionless capture factor $\eta = P/(\mathbb{P}_{pw}\lambda/2\pi)$ against ka when *equal* springs and dampers are optimised to take all the power from a range of circular modes. We have concentrated on piston-like paddle operation with $a/h = 1$, $c/h = \frac{1}{2}$ and used $\mathcal{M} = 0.034\rho ah$, $\mathcal{C} = 0$. Although the theory only guarantees $\eta \geq \epsilon_m$, the rigid-body limit of $\eta = 3$ is exceeded for sufficiently large ka as high absorption happens to take place across other circular modes. The free surface amplitude associated with these results are shown in Fig. 2. Fig. 3 shows the maximum free surface amplitudes when *unequal* springs and dampers are tuned to extract all of the available energy from the first 4 circular modes. It can be seen that paddles must work harder for smaller values of ka to absorb power well in excess of rigid body limits, bringing into question whether this particular embodiment of the general theory is really a practical solution. More extensive results will be shown at the workshop including a comparison between the computations based on the discrete paddle description (10), (11) and the continuum approximation.

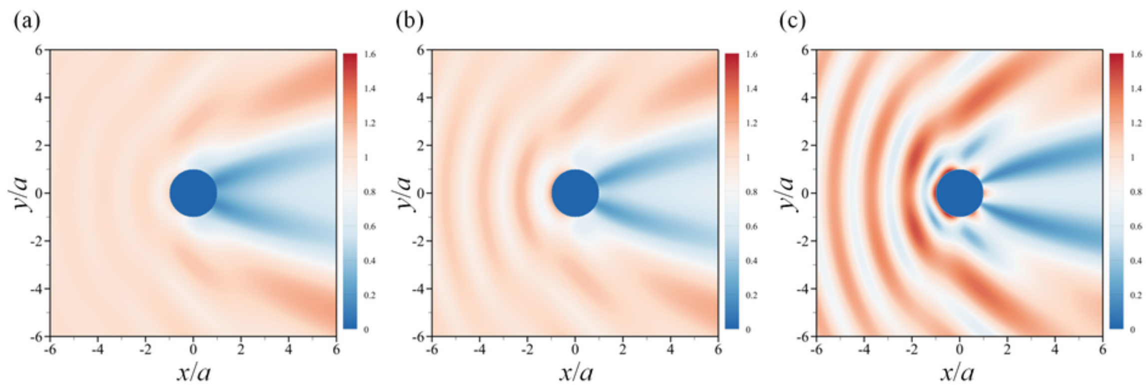

Figure 2: Maximum free surface elevation at $ka = 2$ for equal springs and dampers optimised to absorb 100% of the power from modes: (a) $m = 1$; (b) $m = 2$; (c) $m = 3$.

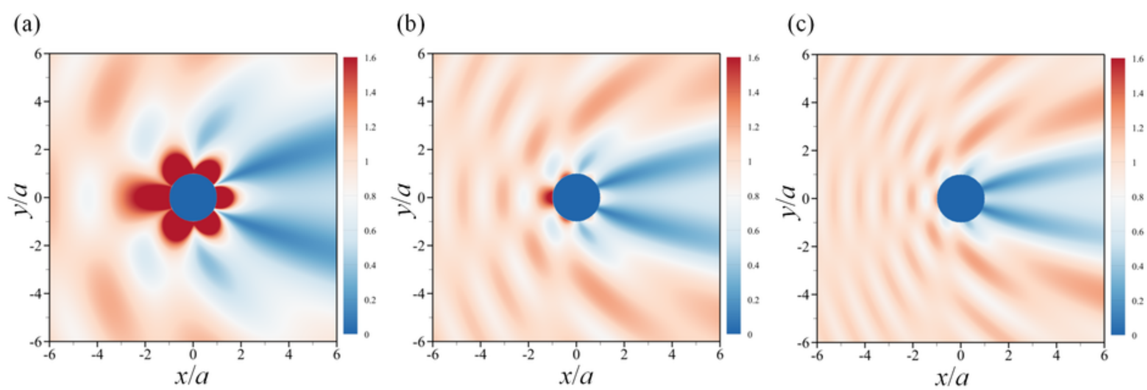

Figure 3: The maximum free surface elevation when unequal springs and dampers are optimised to absorb 100% of the power available from the first 4 modes ($M = 3$, $\eta = 7$) at (a) $ka = 1$, (b) $ka = 2$, (c) $ka = 3$.

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