Example (Non-constant coefficients)

$$x^2 u_{xx} - y^2 u_{yy} = 0, \quad x, y > 0.$$

Here $a = x^2$, $b = 0$ and $c = -y^2$. Hence $b^2 - ac = x^2y^2$ and the eqn is hyperbolic. We have

$$\lambda_\pm = \pm \frac{x y}{x^2} = \pm y/x.$$

$\underline{\lambda = \lambda_-}$: $Y' = \lambda_- = -Y/x$. So $Y'/Y = -dx/x$ and integration yields $Y = c/x$. So we choose $\xi = xy$.

$\underline{\lambda = \lambda_+}$: $Y' = \lambda_+ = Y/x$. So $Y'/Y = dx/x$. So $Y = cx$ and we choose $\eta = y/x$.

Set $u(x,y) = \mathcal{U}(xy, y/x)$. Then

$$u_x = \mathcal{U}_\xi y - \mathcal{U}_\eta y/x^2, \quad u_y = \mathcal{U}_\xi x + \mathcal{U}_\eta \frac{1}{x}$$

$$u_{xx} = \mathcal{U}_{\xi\xi} y^2 - \mathcal{U}_{\xi\eta} y^2/x^2 - \mathcal{U}_{\eta\xi} y^2/x^2 + \mathcal{U}_{\eta\eta} y^2/x^4 + 2\mathcal{U}_\eta y/x^3$$

$$u_{yy} = \mathcal{U}_{\xi\xi} x^2 + \mathcal{U}_{\xi\eta} + \mathcal{U}_{\eta\xi} + \mathcal{U}_{\eta\eta} 1/x^2$$

It follows that

$$0 = x^2 u_{xx} - y^2 u_{yy} = [x^2y^2 - x^2y^2]\,\mathcal{U}_{\xi\xi} + [-2y^2 - 2y^2]\,\mathcal{U}_{\xi\eta} + [y^2/x^2 - y^2/x^2]\,\mathcal{U}_{\eta\eta} + 2y/x\,\mathcal{U}_\eta$$

and, after simplification,

$$\mathcal{U}_{\xi\eta} - \frac{1}{2\xi}\mathcal{U}_\eta = 0.$$

Integrating with respect to η, we obtain

$$\mathcal{U}_\xi - \frac{1}{2\xi}\mathcal{U} = f(\xi)$$

and so

$$\frac{d}{d\xi}\left[\frac{1}{\sqrt{\xi}}\mathcal{U}\right] = f(\xi) \qquad [f \text{ has changed!}]$$

Then $\frac{1}{\sqrt{\xi}}\mathcal{U} = f(\xi) + g(\eta)$ $\qquad$ [f has changed!]

That is: $\mathcal{U} = f(\xi) + \sqrt{\xi}\, g(\eta)$ $\qquad$ [f has changed!]

$$u(x,y) = f(xy) + \sqrt{xy}\; g(y/x).$$

<u>Example</u> (with boundary conditions)

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0$$

subject to

$$u(x,0) = x \quad \text{and} \quad u_y(x,0) = 1. \qquad (C)$$

Here $a=1$, $b=1$ and $c=-3$. The eqn is hyperbolic.

$$\lambda_{\pm} = 1 \pm 2$$

$\underline{\lambda = \lambda_-}$: $Y' = -1$. So $\xi = y + x$.

$\underline{\lambda = \lambda_+}$: $Y' = 3$. So $\xi = y - 3x$.

The canonical form is obviously $u_{\xi\eta} = 0$ and so the general solution

$$u(x,y) = f(x+y) + g(y-3x).$$

We now look for the particular soln satisfying (C). These conditions — expressed in terms of f and g — are

$$(+) \qquad f(x) + g(-3x) = x$$

$$(++) \qquad f'(x) + g'(-3x) = 1$$

The idea is to differentiate (+) with respect to x, so that we get two eqns for the two unknowns f' and g':

$$(+++) \qquad f'(x) - 3g'(-3x) = 1$$

(++) − (+++) gives

$$4g'(-3x) = 0$$

So $g = c = \text{const}$ and, from (+),

$$f(x) = x - c.$$

Reporting this in the formula for u, we obtain

$$u(x,y) = x+y-c+c = x+y.$$

<u>Transforms</u> A linear transform T acting on functions $f : (a,b) \to \mathbb{R}$ or $\mathbb{C}$ can be defined by specifying a <u>kernel</u>

$$K : (a,b) \times (a,b) \to \mathbb{R} \text{ or } \mathbb{C}$$

and using the rule

$$(Tf)(y) = \int_a^b K(x,y)\, f(x)\, dx$$

The maximal domain of T will then depend on the particular choice of K.

<u>Examples</u> (1) $(a,b) = \mathbb{R}$, $K(x,y) = e^{-ixy}$ $\to$ Fourier

(2) $(a,b) = (0,\infty)$, $K(x,y) = e^{-xy}$ $\to$ Laplace

(3) $(a,b) = (0,\infty)$, $K(x,y) = x^{y}$ $\to$ Mellin.

Why are transforms useful? Suppose that f is the unknown soln of a difficult problem (e.g. a differential eqn). It is sometimes possible to find an easier problem for the transform Tf.

$$\begin{tikzcd}
\text{Original problem} \arrow[r, dashed, "\text{difficult}"] \arrow[d, "T"'] & \text{soln } f \\
\text{Transformed problem} \arrow[r, "\text{easy}"'] & \text{soln } Tf \arrow[u, "T^{-1}"']
\end{tikzcd}$$

This strategy only succeeds if the inverse transform may be computed.

In this unit, we will study the Fourier and the Laplace transforms, and discuss their application to partial differential eqns.

<u>The Fourier transform</u> It is standard notation to use k instead of y, and the "hat" $\wedge$ instead of T:

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx .$$

<u>Example</u> Let $f(x) = e^{-a|x|}$ where $a > 0$. Then

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-a|x|-ikx}\, dx = \int_{-\infty}^{0} e^{ax-ikx}\, dx + \int_{0}^{\infty} e^{-ax-ikx}\, dx$$

$$= \frac{1}{a-ik} e^{(a-ik)x}\Big|_{-\infty}^{0} - \frac{1}{a+ik} e^{-(a+ik)x}\Big|_{0}^{\infty}$$

$$= \frac{1}{a-ik} + \frac{1}{a+ik} = \frac{2a}{a^2+k^2} .$$

<u>Example</u> $f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{otherwise} \end{cases}$.

Then

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx = \int_{-a}^{a} e^{-ikx}\, dx = \frac{1}{-ik} e^{-ikx}\Big|_{-a}^{a}$$

$$= \frac{e^{-ika} - e^{ika}}{-ik} = \frac{2}{k} \frac{e^{ika} - e^{-ika}}{2i} = \frac{2}{k}\sin(ka)$$

<u>The inversion formula</u> We shall use without proof the fact that

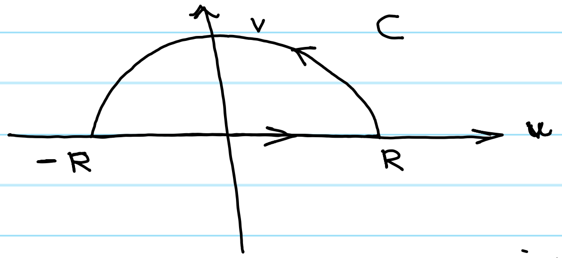
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk$$

<u>Remark</u> This formula says that f is the complex conjugate of the Fourier transform of $\overline{\hat{f}(k)/2\pi}$. [The bar denotes complex conjugation.].

<u>Example</u> We saw in a previous example that $f(x) = e^{-a|x|}$ has the transform $\hat{f}(k) = \frac{2a}{a^2+k^2}$

Let us compute $\displaystyle\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk = \int_{-\infty}^{\infty} \frac{2a}{k^2+a^2}\, e^{ikx}\, dk$

We use contour integration : set $F(w) = \dfrac{2a}{w^2+a^2}\, e^{iwx}$ where $w = u + iv$ and consider the contour



Here we consider the case $x > 0$.

For R large enough, the contour C will enclose a pole of F — the pole at $w = ia$. The residue at that pole is

$$\lim_{w \to ia} (w - ia)\, F(w) = \lim_{w \to ia} \frac{2a\, e^{iwx}}{w + ia} = \frac{2a\, e^{-ax}}{2ia}$$

There is no other pole inside the contour and so, by the residue theorem

$$2\pi i \, \frac{2a e^{-ax}}{2ia} = \int_C F(w)\, dw = \int_{-R}^{R} \frac{2a\, e^{ixu}}{a^2+u^2}\, du$$

$$+ \int_0^{\pi} \frac{2a\, e^{ixRe^{i\theta}}}{a^2 + R^2 e^{i2\theta}}\, iRe^{i\theta}\, d\theta$$

We now let $R \to \infty$. The second integral on the right-hand side tends to zero in that limit because

$$e^{ixRe^{i\theta}} = e^{ixR\cos\theta}\, e^{-xR\sin\theta} \xrightarrow{R \to \infty} 0$$

[We have assumed $x > 0$ and chosen C so that $\sin\theta > 0$]

Hence

$$\cancel{2\pi e^{-ax}} \quad 2\pi\, e^{-ax} = \int_{-\infty}^{\infty} \hat{f}(u)\, e^{ixu}\, du$$

and so the inversion formula is verified for $x > 0$. For $x < 0$, we need to choose the contour obtained by reflecting about the u axis.

Properties of the Fourier transform

Linearity: $\widehat{f+g} = \hat{f} + \hat{g}$ and $\widehat{\lambda f} = \lambda \hat{f}$

scaling Set $g(x) = f(\alpha x)$ where $\alpha > 0$. Then
$$\hat{g}(k) = \frac{1}{\alpha} \hat{f}\left(\frac{k}{\alpha}\right).$$

translation Set $g(x) = f(x+a)$. Then
$$\hat{g}(k) = e^{ika} \hat{f}(k)$$
Likewise, set $g(x) = e^{ax} f(x)$. Then
$$\hat{g}(k) = \hat{f}(k+ia)$$

Differentiation: $\frac{d}{dk} \hat{f}(k) = \int_{-\infty}^{\infty} (-ix) f(x) e^{-ikx} \, dx$

Hence
$$\widehat{xf} = i \hat{f}'(k)$$
Likewise
$$\int_{-\infty}^{\infty} f'(x) e^{-ikx} \, dx = \underbrace{\cancel{f(x) e^{-ikx} \Big|_{-\infty}^{\infty}}}_{0 \text{ if } |f| \text{ is integrable.}} + \int_{-\infty}^{\infty} ik f(x) e^{-ikx} \, dx$$
That is: $\widehat{f'} = ik \hat{f}$.

Definition A function $f : \mathbb{R} \to \mathbb{C}$ is called <u>square-integrable</u> if
$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$$

Definition The <u>convolution</u> $f * g$ of two square-integrable functions f and g is the function defined by
$$(f*g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) \, dy$$

Remark $(g*f)(x) = \int_{-\infty}^{\infty} g(y) f(x-y) \, dy \overset{t = x-y}{=} \int_{-\infty}^{\infty} g(x-t) f(t) \, dt$
$$= (f*g)(x)$$
So the order of f and g does not matter.

The convolution theorem says that

$$\widehat{f*g} = \hat{f}\,\hat{g}$$

Proof: $\widehat{f*g}(k) = \int_{-\infty}^{\infty} (f*g)(x)\, e^{-ikx}\, dx$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \right\} e^{-ikx}\, dx$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(y)\, e^{-iky}\, g(x-y)\, e^{-ik(x-y)}\, dy \right\} dx$$

$$= \int_{-\infty}^{\infty} f(y)\, e^{-iky} \left\{ \int_{-\infty}^{\infty} g(x-y)\, e^{-ik(x-y)}\, dx \right\} dy$$

Make the substitution $t = x-y$ in the inner integral:

$$\widehat{f*g}(k) = \int_{-\infty}^{\infty} f(y)\, e^{-iky} \left\{ \int_{-\infty}^{\infty} g(t)\, e^{-ikt}\, dt \right\} dy$$

$$= \int_{-\infty}^{\infty} f(y)\, e^{-iky}\, dy \int_{-\infty}^{\infty} g(t)\, e^{-ikt}\, dt$$

$$= \hat{f}(k)\, \hat{g}(k) \qquad \blacksquare$$

Parseval's identity says

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(k)|^2\, dk$$

Proof: Let f and g be square-integrable, so that $f*g$ is defined. The convolution theorem says

$$\widehat{f*g} = \hat{f}\,\hat{g}$$

So, by the inversion formula,

$$(f*g)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\, \hat{g}(k)\, e^{ikx}\, dk \qquad (*)$$

~~Equivalently~~: Now, make the specific choice

$$g(y) = \overline{f(-y)}$$

~~Then (*) becomes~~ With this choice

$$\cancel{\frac{1}{2\pi}} \cdot \hat{g}(k) = \int_{-\infty}^{\infty} \overline{f(-y)}\, e^{-iky}\, dy \overset{x=-y}{=} \int_{-\infty}^{\infty} \overline{f(x)}\, e^{ikx}\, dx$$

$$= \overline{\int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx} = \overline{\hat{f}(k)}.$$

So $(*)$ becomes, after setting $x=0$,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) \overline{\hat{f}(k)} \, dk = \int_{-\infty}^{\infty} f(y) \overline{f(y)} \, dy \qquad \blacksquare$$

Example For $f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{otherwise} \end{cases}$ we found earlier

$\hat{f}(k) = \frac{2}{k} \sin(ka)$. Parseval's identity says

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{k^2} \sin^2(ka) \, dk = 2a$$