Application to the heat eqn

Consider

$$\theta_t = \sigma^2 \theta_{xx}, \quad x \in \mathbb{R}, t > 0, \qquad (H)$$

subject to

$$\theta(x,0) = f(x), \quad x \in \mathbb{R}. \qquad (C)$$

θ is interpreted as the temperature along an infinite rod.

Solution: Here, we look for a solution s.t. the Fourier transform

$$\hat{\theta}(k,t) = \int_{-\infty}^{\infty} \theta(x,t)\, e^{-ikx}\, dx$$

exists. Multiply the heat eqn by e^{-ikx} and integrate wrt x:

$$\int_{-\infty}^{\infty} \theta_t(x,t)\, e^{-ikx}\, dx = \sigma^2 \int_{-\infty}^{\infty} \theta_{xx}(x,t)\, e^{-ikx}\, dx.$$

For the left-hand side, we have

$$\int_{-\infty}^{\infty} \theta_t(x,t)\, e^{-ikx}\, dx = \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \theta(x,t)\, e^{-ikx}\, dx = \frac{\partial}{\partial t} \hat{\theta}(k,t)$$

For the right-hand side:

$$\int_{-\infty}^{\infty} \theta_{xx}(x,t)\, e^{-ikx}\, dx = \theta_x(x,t)\, e^{-ikx}\Big|_{-\infty}^{\infty} + ik \int_{-\infty}^{\infty} \theta_x(x,t)\, e^{-ikx}\, dx$$

$$= \theta_x(x,t)\, e^{-ikx}\Big|_{-\infty}^{\infty} + ik\theta(x,t)\, e^{-ikx}\Big|_{-\infty}^{\infty} + i^2k^2 \int_{-\infty}^{\infty} \theta(x,t)\, e^{-ikx}\, dx.$$

We have to decide what to do with the boundary terms: we shall suppose that θ and θ_x decay as $|x| \to \infty$ (to zero). Then

$$\int_{-\infty}^{\infty} \theta_{xx}(x,t)\, e^{-ikx}\, dx = -k^2 \int_{-\infty}^{\infty} \theta(x,t)\, e^{-ikx}\, dx = -k^2\, \hat{\theta}(k,t).$$

So we obtain from the heat eqn the following eqn for $\hat{\theta}$:

$$\frac{\partial \hat{\theta}}{\partial t} = -\sigma^2 k^2\, \hat{\theta} \qquad (TH)$$

Furthermore, the initial condition, i.e. $\theta(x,0) = f(x)$, obviously translates into

$$\hat{\theta}(k,0) = \hat{f}(k). \qquad (\mathrm{TC})$$

We observe that the problem (TH)$-$(TC) is much easier to solve than the original problem (H)$-$(C). Indeed

$$\hat{\theta}(k,t) = \hat{f}(k) \underbrace{\exp\left[-\sigma^2 k^2 t\right]}_{\hat{g}(k|t)}$$

The right-hand side is the product of two transforms. Hence $\hat{\theta}$ is the transform of the convolution

$$\theta(x,t) = \int_{-\infty}^{\infty} f(x-y)\, g(y,t)\, dy \qquad (\mathrm{S})$$

where, by the inversion formula

$$g(y,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[-\sigma^2 k^2 t + iky\right] dk$$

Now

$$\exp\left[-\sigma^2 k^2 t + iky\right] = e^{-y^2/4\sigma^2 t} \exp\left[-\sigma^2 t\left(k - \frac{iy}{2\sigma^2 t}\right)^2\right]$$

and so

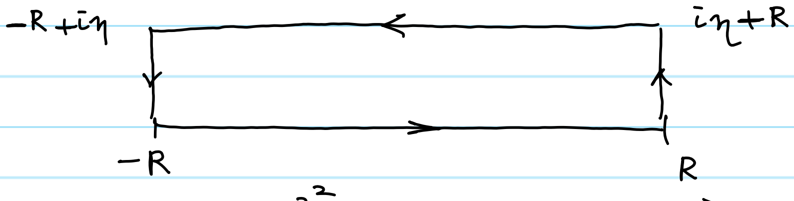
$$g(y,t) = \frac{1}{2\pi} e^{-\frac{y^2}{4\sigma^2 t}} \int_{-\infty}^{\infty} \exp\left[-\sigma^2 t\left(k - \frac{iy}{2\sigma^2 t}\right)^2\right] dk$$

$$\underset{u = k - \frac{iy}{2\sigma^2 t}}{\overset{\uparrow}{=}} \frac{1}{2\pi} e^{-\frac{y^2}{4\sigma^2 t}} \int_{-\infty - \frac{iy}{2\sigma^2 t}}^{\infty - \frac{iy}{2\sigma^2 t}} \exp\left[-\sigma^2 t u^2\right] du$$

<u>Aside</u> Let us argue that, for $\eta \in \mathbb{R}$,

$$\int_{-\infty+i\eta}^{\infty+i\eta} e^{-x^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, dx$$

Why? Suppose for instance that $\eta > 0$ and consider the contour



The function e^{-z^2} has no pole inside this contour. So

$$0 = \int_C e^{-z^2}\,dz = \int_{-R}^{R} e^{-x^2}\,dx + \int_0^{\eta} e^{-(R+iy)^2}\, i\,dy$$

$$+ \int_R^{-R} e^{-(x+i\eta)^2}\,dx + \int_{\eta}^{0} e^{-(-R+iy)^2}\, i\,dy$$

$$\xrightarrow[R\to\infty]{} \int_{-\infty}^{\infty} e^{-x^2}\,dx + \underbrace{\int_{\infty}^{-\infty} e^{-(x+i\eta)^2}\,dx}_{\underset{y=x+i\eta}{=} \int_{\infty+i\eta}^{-\infty+i\eta} e^{-y^2}\,dy}$$

Hence

$$\int_{-\infty+i\eta}^{\infty+i\eta} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

<u>End of aside</u>

So, applying this to our situation, we obtain

$$g(y,t) = \frac{1}{2\pi} \exp\left[-\frac{y^2}{4\sigma^2 t}\right] \int_{-\infty}^{\infty} \exp\left[-\sigma^2 t u^2\right] du$$

$$\underset{\xi = \sigma\sqrt{t}\,u}{=} \frac{1}{2\pi}\,\frac{1}{\sigma\sqrt{t}} \exp\left[\frac{-y^2}{4\sigma^2 t}\right] \underbrace{\int_{-\infty}^{\infty} \exp(-\xi^2)\,d\xi}_{=\sqrt{\pi}}$$

So

$$g(y,t) = \frac{1}{2\sigma\sqrt{\pi t}} \exp\left[-\frac{y^2}{4\sigma^2 t}\right]$$

Reporting this in (5) on p.2:

$$\theta(x,t) = \frac{1}{2\sigma\sqrt{\pi t}} \int_{-\infty}^{\infty} f(x-y) \exp\left[-\frac{y^2}{4\sigma^2 t}\right] dy$$

<u>Example</u> Suppose $f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{otherwise} \end{cases}$.

Then

$$\theta(x,t) = \frac{1}{2\sigma\sqrt{\pi t}} \int_{x-a}^{x+a} \exp\left[-\frac{y^2}{4\sigma^2 t}\right] dy \qquad (*)$$

This can be expressed more neatly in terms of the error function

$$\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

Indeed, with the substitution $\frac{y}{2\sigma\sqrt{t}} = \xi$ in $(*)$:

$$\theta(x,t) = \frac{1}{\sqrt{\pi}} \left\{ \int_0^{\frac{x+a}{2\sigma\sqrt{t}}} e^{-\xi^2} d\xi - \int_0^{\frac{x-a}{2\sigma\sqrt{t}}} e^{-\xi^2} d\xi \right\}$$

$$= \frac{1}{2} \left\{ \text{erf}\left(\frac{x+a}{2\sigma\sqrt{t}}\right) - \text{erf}\left(\frac{x-a}{2\sigma\sqrt{t}}\right) \right\}.$$

<u>Sine and cosine transforms</u>

Suppose we are given a function defined only for $x > 0$. We can choose to extend it in many ways. We consider two of them.

(1) even extension: for $x < 0$,

$$f(x) = +f(-x).$$

Then

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_{-\infty}^{\infty} f(x) [\cos(kx) - i\sin(kx)] dx$$

$$= 2\int_0^{\infty} f(x) \cos(kx) dx \qquad [\text{since } f \text{ is even}].$$

The inversion formula gives

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$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk = \frac{1}{\pi}\int_0^{\infty} \hat{f}(k)\cos(kx)\, dk$$

since $\hat{f}$ is even. This suggests defining the cosine transform of $f : [0,\infty) \to \mathbb{R}$ by

$$\hat{f}_c(k) = \int_0^{\infty} f(x)\cos(kx)\, dx$$

Then

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \hat{f}_c(k)\cos(kx)\, dk$$

(2) odd extension : for $x < 0$,

$$f(x) = -f(-x).$$

Then

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)\left[\cos(kx) - i\sin(kx)\right] dx$$

$$= -2i\int_0^{\infty} f(x)\sin(kx)\, dx \qquad [\text{since } f \text{ is odd}]$$

and the inversion formula gives

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk = \frac{i}{\pi}\int_0^{\infty} \hat{f}(k)\sin(kx)\, dk$$

since $\hat{f}$ is odd. This suggests defining the sine transform by

$$\hat{f}_s(k) = \int_0^{\infty} f(x)\sin(kx)\, dx.$$

Then

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \hat{f}_s(k)\sin(kx)\, dk.$$

<u>Example</u> Let

$$f(x) = e^{-bx}, \quad x > 0,\ b > 0.$$

Then

$$\hat{f}_c(k) = \int_0^{\infty} e^{-bx}\cos(kx)\, dx = -\frac{1}{b}e^{-bx}\cos(kx)\Big|_0^{\infty}$$

$$- \frac{k}{b}\int_0^{\infty} e^{-bx}\sin(kx)\, dx$$

$$= \frac{1}{b} - \frac{k}{b}\left\{ -\frac{1}{b} e^{-bx}\sin(kx)\Big|_0^\infty + \frac{k}{b}\int_0^\infty e^{-bx}\cos(kx)\,dx \right\}$$

$$= \frac{1}{b} - \frac{k^2}{b^2}\hat{f}_c(k).$$

So $\hat{f}_c(k) = \dfrac{1/b}{1+k^2/b^2} = \dfrac{b}{b^2+k^2}$.

The inversion formula then says

$$e^{-bx} = \frac{2}{\pi}\int_0^\infty \frac{b\cos(kx)}{k^2+b^2}\,dk$$

A similar calculation for $\hat{f}_s$ yields

$$\hat{f}_s(k) = \frac{k}{k^2+b^2} \quad \text{and} \quad e^{-bx} = \frac{2}{\pi}\int_0^\infty \frac{k\sin(kx)}{k^2+b^2}\,dk$$

<u>Example</u> We revisit our heat conduction problem — this time for a semi infinite rod:

$$u_t = \sigma^2 u_{xx}, \quad x>0,\ t>0$$

s.t.

$$u(x,0)=0,\ x>0 \quad \text{and} \quad u(0,t) = u_0 = \text{const},\ t>0.$$

Here we have a choice of two transforms; sine and cosine. It turns out that the particular form of the boundary condition makes the sine transform more convenient. We therefore seek an eqn for $\hat{u}_s(k,t)$. Multiply the heat eqn by $\sin(kx)$ and integrate over x:

$$\frac{\partial}{\partial t}\hat{u}_s(k,t) = \sigma^2\int_0^\infty u_{xx}(x,t)\sin(kx)\,dx$$

$$= \sigma^2 \cancelto{0}{u_x(x,t)\sin(kx)\Big|_0^\infty} - k\sigma^2\int_0^\infty u_x(x,t)\cos(kx)\,dx$$

$$= -k\sigma^2 u(x,t)\cos(kx)\Big|_0^\infty - k^2\sigma^2\int_0^\infty u(x,t)\sin(kx)\,dx$$

$$= k\sigma^2 u_0 - k^2\sigma^2\hat{u}_s(k,t).$$

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So the eqn for ~~ϕ~~ $\hat{u}_s$ is

$$\frac{\partial \hat{u}_s(k,t)}{\partial t} = -k^2\sigma^2 \hat{u}_s(k,t) + k\sigma^2 u_0$$

s.t. $\hat{u}_s(k,0) = 0$. Using the obvious integrating factor, we can write

$$\frac{\partial}{\partial t}\left\{ \exp\left[k^2\sigma^2 t\right] \hat{u}_s(k,t) \right\} = k\sigma^2 u_0 \exp\left[k^2\sigma^2 t\right]$$

and so

$$\exp\left[k^2\sigma^2 t\right] \hat{u}_s(k,t) - \hat{u}_s(k,0) = k\sigma^2 u_0 \int_0^t \exp\left[k^2\sigma^2\tau\right] d\tau$$

$$= \frac{u_0}{k} \exp\left[k^2\sigma^2\tau\right]\Big|_0^t = \frac{u_0}{k}\left\{ e^{k^2\sigma^2 t} - 1 \right\}$$

That is:

$$\hat{u}_s(k,t) = \frac{u_0}{k}\left(1 - e^{-k^2\sigma^2 t}\right)$$

There remains to invert the transform:

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{u_0}{k}\left(1 - e^{-k^2\sigma^2 t}\right)\sin(kx)\, dk$$

$$= \frac{2u_0}{\pi}\int_0^\infty \frac{\sin(kx)}{k}\, dk - \frac{2u_0}{\pi}\int_0^\infty e^{-k^2\sigma^2 t}\frac{\sin(kx)}{k}\, dk$$

From Sheet 1: $\int_0^\infty \frac{\sin(kx)}{k}\, dk = \pi/2$. So

$$u(x,t) = u_0\left\{ 1 - \varphi(x,t) \right\} \qquad (+)$$

where

$$\varphi(x,t) = \frac{2}{\pi}\int_0^\infty e^{-k^2\sigma^2 t}\frac{\sin(kx)}{k}\, dk$$

We note that $\varphi(0,t) = 0$ and

$$\varphi_x(x,t) = \frac{2}{\pi}\int_0^\infty e^{-k^2\sigma^2 t}\cos(kx)\, dk$$

$$= \frac{1}{\pi}\int_0^\infty e^{-k^2\sigma^2 t}\left[e^{ikx} + e^{-ikx}\right] dk$$

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$$= \frac{1}{\pi}\left\{ \int_0^\infty e^{-\sigma^2 k^2 t + ikx}\, dk + \int_{-\infty}^0 e^{-\sigma^2 k^2 t + ikx}\, dk \right\}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \exp\left[-\sigma^2 t\left(k^2 - \frac{ikx}{\sigma^2 t}\right)\right] dk$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \exp\left[-\sigma^2 t\left(k - \frac{ix}{2\sigma^2 t}\right)^2\right] \exp\left[-\frac{x^2}{4\sigma^2 t}\right] dk$$

$$\underset{\xi = k - \frac{ix}{2\sigma^2 t}}{\overset{\uparrow}{=}} \exp\left[-\frac{x^2}{4\sigma^2 t}\right] \frac{1}{\pi}\int_{-\infty}^{\infty} \exp\left[-\sigma^2 t \xi^2\right] d\xi$$

$$= \exp\left[-\frac{x^2}{4\sigma^2 t}\right] \frac{1}{\sigma\sqrt{\pi t}}$$

So we have found

$$\varphi_x(x,t) = \frac{1}{\sigma\sqrt{\pi t}} \exp\left[-\frac{x^2}{4\sigma^2 t}\right]$$

Using $\varphi(0,t) = 0$ and integrating, this gives

$$\varphi(x,t) = \frac{1}{\sigma\sqrt{\pi t}} \int_0^x \exp\left[-\frac{\xi^2}{4\sigma^2 t}\right] d\xi$$

$$\underset{\xi = \frac{y}{2\sigma\sqrt{t}}}{\overset{\uparrow}{=}} \frac{2}{\sqrt{\pi}} \int_0^{x/2\sigma\sqrt{t}} e^{-y^2}\, dy = \operatorname{erf}\left(\frac{x}{2\sigma\sqrt{t}}\right).$$

We conclude from (+), p. 7 that

$$u(x,t) = u_0\left[1 - \operatorname{erf}\left(\frac{x}{2\sigma\sqrt{t}}\right)\right] = u_0 \operatorname{erfc}\left(\frac{x}{2\sigma\sqrt{t}}\right).$$

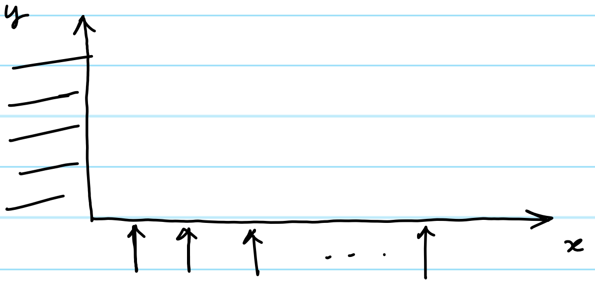
<u>Note</u>: The <u>error function</u> erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy$$

and the <u>complementary error function</u> erfc by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x).$$

Example : Two-dimensional potential flow driven by a source



$x, y > 0$

The fluid's velocity field is the gradient of a scalar potential :

$$\underline{u} = \nabla\varphi = (\varphi_x, \varphi_y)$$

and the fluid is assumed to be incompressible, i.e. $\operatorname{div} \underline{u} = 0$. Hence

$$\varphi_{xx} + \varphi_{yy} = 0 \quad , \quad x, y > 0 \qquad (E)$$

It is assumed that the vertical wall at $x=0$ is impermeable, i.e. the horizontal component of the velocity vanishes there

$$\varphi_x(0,y) = 0 \quad , \quad y > 0 \ . \qquad (B)$$

Fluid is pumped through the horizontal floor at $y=0$, so that the vertical component of the velocity is prescribed there

$$\varphi_y(x,0) = g(x) \quad , \quad x > 0 \qquad (C)$$

In addition, since we wish to use transforms, it will be assumed that φ and its derivatives ~~vanish~~ decay to zero for $x, \cancel{y}$ large. We will also assume

$$\underline{u} \cdot \xrightarrow[y \to \infty]{} 0 \qquad (\infty)$$

The form of the ~~bounda~~ boundary condition (B) suggests using the cosine transform

$$\hat{\varphi}_c(k,y) = \int_0^\infty \varphi(x,y)\cos(kx)\,dx$$

Multiply (E) by $\cos(kx)$ and integrate over x :

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$$\frac{\partial^2 \hat{\varphi}_c(k,y)}{\partial y^2} = -\int_0^\infty \varphi_{xx}(x,y)\cos(kx)\,dx$$

$$= -\varphi_x(x,y)\cos(kx)\Big|_0^\infty - k\int_0^\infty \varphi_x(x,y)\sin(kx)\,dx$$

$$= -\varphi_x(x,y)\cos(kx)\Big|_0^\infty - k\,\varphi(x,y)\sin(kx)\Big|_0^\infty + k^2\int_0^\infty \varphi(x,y)\cos(kx)\,dx$$

We look for a soln such that

$$\varphi_x(x,y) \text{ and } \varphi(x,y) \xrightarrow[x\to\infty]{} 0, \quad y>0$$

Then

$$\frac{\partial^2}{\partial y^2}\hat{\varphi}_c(k,y) = k^2\hat{\varphi}_c(k,y) \qquad \text{(TE)}$$

s.t. $$\frac{\partial \hat{\varphi}_c}{\partial y}(k,0) = \hat{g}_c(k) \qquad \text{(TC)}$$

The general soln of (TE) is

$$\hat{\varphi}_c(k,y) = A(k)e^{ky} + B(k)e^{-ky}$$

Now, the condition (∞) implies in particular that

$$\varphi_y(x,y) \xrightarrow[y\to\infty]{} 0$$

Therefore $\frac{\partial}{\partial y}\hat{\varphi}_c(k,y) \xrightarrow[y\to\infty]{} 0$ and it follows that

$$A(k) = 0$$

Hence

$$\hat{\varphi}_c(k,y) = B(k)e^{-ky} \overset{\text{(TC)}}{=} -\frac{\hat{g}_c(k)}{k}e^{-ky}$$

The inversion formula then yields

$$\varphi(x,y) = -\frac{2}{\pi}\int_0^\infty \hat{g}_c(k)\,e^{-ky}\,\cancel{\sin}\,\frac{\cos(kx)}{k}\,dk$$

This is as far as one can go.