Math 30800 Week 1

Characteristics are curves associated with PDEs that can be used to express the soln in terms of the data.

We begin our study with a simple example:

$$u_t = u_x \,, \quad x \in \mathbb{R},\ t > 0 \qquad (E)$$

s.t.

$$u(x,0) = f(x)$$

Introduce an arbitrary curve of eqn

$$x = X(t) \,, \quad t > 0$$

NOTATION $\dot{X} = \frac{dX}{dt}$

Then

$$\frac{d}{dt} u(X(t),t) = u_x(X(t),t)\,\dot{X} + u_t(X(t),t)$$

Now, suppose that the curve satisfies $\dot{X} = 1$, i.e.

$$X(t) = \xi + t \qquad (C)$$

Then

$$\frac{d}{dt} u(X(t),t) = 0 \qquad [\text{since } u \text{ solves } (E)]$$

So the function $t \longmapsto u(X(t),t)$ is constant, i.e.

$$u(X(t),t) = u(X(0),0) = u(\xi,0) = f(\xi)$$

This holds for every ξ and every $t > 0$. By choosing $\xi = x - t$, we deduce

$$u(x,t) = f(x-t)$$

and we have solved our problem! The curves (C) are called the characteristic curves associated with (E).

In what follows, we shall develop this method up & build on so that it may be used for the more general !!

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$$a u_x + b u_y = c$$

where a, b and c may depend on x, y and u.

We suppose that the soln is prescribed along some curve

$$x = x_0(\xi), \quad y = y_0(\xi), \quad u = u_0(\xi)$$

called the <u>initial curve</u>. To solve the problem, we examine how the soln varies along an arbitrary curve of parametric form

$$x = X(s), \quad y = Y(s)$$

We have

$$\frac{d}{ds} u(X(s), Y(s)) = u_x(X, Y)\dot{X} + u_y(X, Y)\dot{Y}$$

[Notation: $\dot{X} = \frac{d}{ds} X$].

In particular, along the curves s.t.

$$\dot{X} = a, \quad \dot{Y} = b \qquad (*)$$

we obtain

$$\frac{d}{ds} u(X, Y) = c \qquad (D)$$

The curves satisfying $(*)$ are called <u>characteristics</u>. The interest of characteristics is that (D) is an <u>ordinary</u> differential eqn.

Since $(*)$ only specifies the derivative of X and Y, we can impose the initial condition

$$X(0) = x_0(\xi) \quad \text{and} \quad Y(0) = y_0(\xi)$$

This makes it clear that X and Y depend on ξ and s.

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This is an important point, to which we shall return later.

Integrate both sides of (D) with respect to the parameter s:

$$u(X(s),Y(s)) = u(X(0),Y(0)) + \int_0^s c(X(t),Y(t),u(X(t),Y(t)))\,dt$$

$$= u_0(\xi) + \int_0^s c(X(t),Y(t),u(X(t),Y(t)))\,dt \qquad (S)$$

In particular, when c does not depend on u, the integral on the right can in principle be evaluated and so we obtain a formula for $u(X(s),Y(s))$ in terms of the data. In order to obtain a formula for $u(x,y)$ where x and y are arbitrary we need to solve

$$x = X(s,\xi) \quad \text{and} \quad y = Y(s,\xi) \qquad (**)$$

[Here the notation emphasises the already-stated fact that the characteristics depend on s AND ξ]

The eqn $(**)$ is to be solved for s and ξ, and the result is then reported in (S).

<u>Example 1</u> Consider

$$u_x = e^y u_y \quad \text{s.t.} \quad u(0,y) = \operatorname{ch} y$$

Here

$$x_0(\xi) = 0, \quad y_0(\xi) = \xi, \quad u_0(\xi) = \operatorname{ch}\xi$$

The characteristics satisfy

$$\frac{dX}{ds} = 1 \quad \text{and} \quad \frac{dY}{ds} = -e^Y$$

Hence $X = s$ and $Y = -\ln(s + e^{-\xi})$. Along this characteristic (S) yields

$$u(s, -\ln(s+e^{-\xi})) = \operatorname{ch}\xi \qquad [\text{since } c=0]$$

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To find a formula for $u(x,y)$ we need to solve

$$x = s \quad \text{and} \quad y = -\ln(s + e^{-\xi})$$

for the unknowns s and ξ. Obviously

$$s = x \quad \text{and} \quad e^{-\xi} = e^{-y} - x$$

Hence

$$\cancel{u(x,y)} \quad u(x,y) = ch\,\xi = \frac{1}{2} e^{\xi} + \frac{1}{2} e^{-\xi}$$

$$= \frac{1}{2}\left\{ \frac{1}{e^{-y} - x} + e^{-y} - x \right\}$$

<u>Example 2</u> Consider

$$u_x + 2u_y = y e^{x} \quad \text{s.t.}$$

$$u = \sin x \quad \text{along the straight line } y = x .$$

Here $x_0(\xi) = \xi$, $y_0(\xi) = \xi$, $u_0(\xi) = \sin \xi$.

The characteristics are

$$X = s + \xi \quad , \quad Y = 2s + \xi$$

and (S) says

$$u(s+\xi, 2s+\xi) = \sin\xi + \int_0^s (2t + \xi)\, e^{t+\xi}\, dt$$

$$= \sin\xi + e^{\xi}\left\{ (2s+\xi)\, e^{s} - \xi - 2(e^{s} - 1) \right\}$$

We then need to solve

$$x = s + \xi \quad \text{and} \quad y = 2s + \xi$$

with respect to s and ξ :

$$s = y - x \quad \text{and} \quad \xi = 2x - y$$

Hence

$$u(x,y) = \sin(2x - y) + e^{2x-y}\left\{ y + 2 - 2x \right\} + (y-2)\, e^{x}$$

<u>Remark</u> The mapping between (s,ξ) and (X,Y) is invertible iff the jacobian of the mapping does not vanish.

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$$0 \neq \begin{vmatrix} X_s & X_\xi \\ Y_s & Y_\xi \end{vmatrix} = X_s Y_\xi - X_\xi Y_s = a Y_\xi - b X_\xi$$

[since $X_s = a$ and $Y_s = b$].

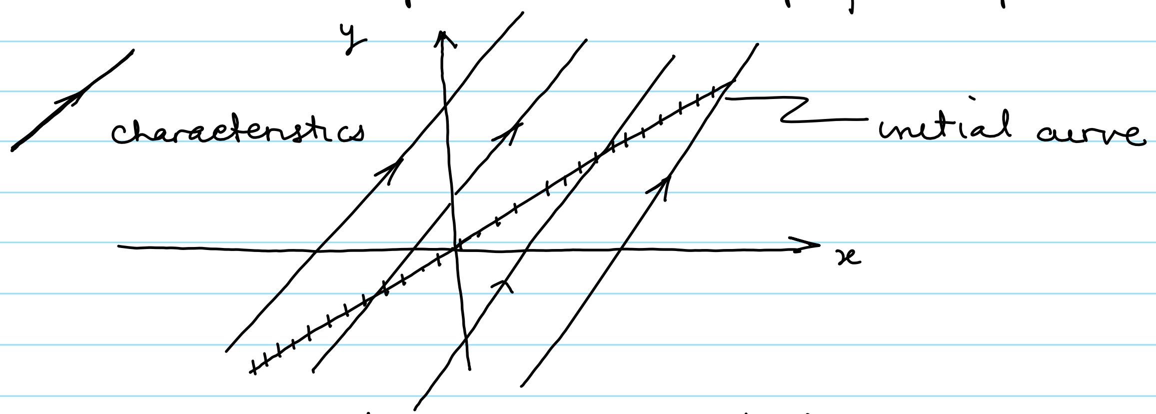
In particular, for $s=0$, we have $Y_\xi = y_0'(\xi)$ and $X_\xi = x_0'(\xi)$
Hence the method is successful iff

$$a y_0' \neq b x_0'$$

Equivalently $\dfrac{dy_0}{dx_0} = b/a$.

This says that the method succeeds provided that the initial curve is NOT a characteristic.

We can illustrate this point with the help of Example 2:



The problem is solvable because the initial curve is not a characteristic.

Remark If the data are discontinuous, the solution will also have discontinuities. Consider for example

$$u_t + u_x = 0$$

s.t.

$$u(x,0) = u_0(x) := \begin{cases} x & \text{if } 0 < x < 1 \\ 2(2-x) & \text{if } 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

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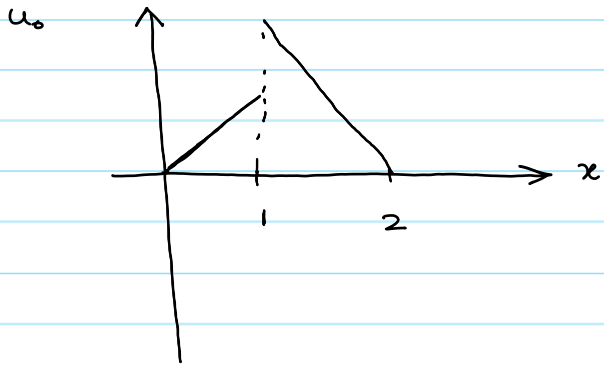
A plot of u_0 against x is shown below:



The soln is given by $u(x,t) = u_0(x-t)$. The discontinuity propagates at speed 1.

<u>The inviscid Burgers eqn</u> $\quad u_t + uu_x = 0$ s.t. $u(x,0) = u_0(x)$

~~The initial curve is parametrised as $x_0(0) = \xi$~~

The characteristics are given by

$$\frac{dX}{dt} = u(X,t), \quad X(0) = \xi \qquad (C)$$

Then $\frac{d}{dt} u(X(t),t) = 0$ and so

$$u(X(t),t) = u(X(0),0) = u_0(\xi)$$

Reporting this in (C), we obtain

$$\frac{dX}{dt} = u_0(\xi), \quad X(0) = \xi$$

and so

$$X(t) = u_0(\xi)t + \xi \qquad (K)$$

These are the characteristic curves associated with this problem.
Hence

$$u(\xi + u_0(\xi)t, t) = u_0(\xi) \qquad (S)$$

To find an expression for $u(x,t)$ for arbitrary x and t, we need to solve the eqn

$$x = u_0(\xi)t + \xi \qquad (E)$$

for ξ.

<u>Examples</u> (1) Take $u_0(x) = x$ as initial datum. Then (E) becomes

$$x = \xi t + \xi$$

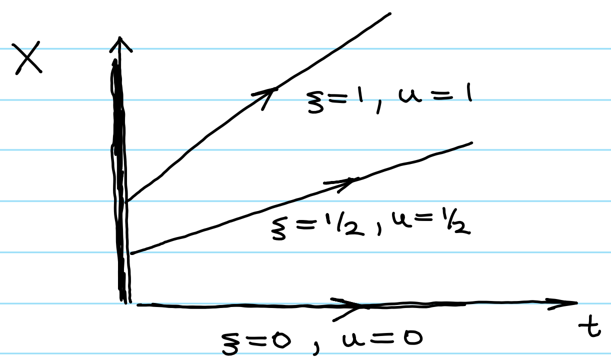
and so

$$\xi = \frac{x}{1+t}$$

Substitution in (S) yields the formula

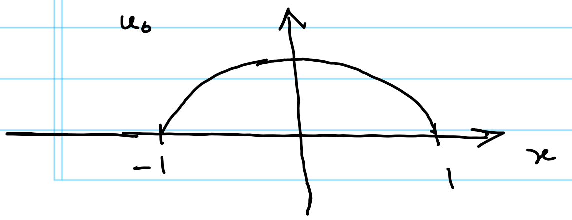
$$u(x,t) = u_0\left(\frac{x}{1+t}\right) = \frac{x}{1+t}$$

You can check that this does indeed solve the problem. It is instructive to draw some of the characteristic curves for this datum:



On each curve, the solution is constant. Furthermore, the curves do not intersect.

(2) Next take $u_0(x) = \begin{cases} 1-x^2 & \text{if } |x|<1 \\ 0 & \text{otherwise} \end{cases}$



To obtain a formula for $u(x,t)$ from (5), we need to solve

$$x = \xi + t(1-\xi^2) \quad , \quad |\xi| < 1 \qquad (Q)$$

This gives

$$\xi = \frac{1 \pm \sqrt{1-4t(x-t)}}{2t}$$

Which sign? Remark that, by the binomial expansion,

$$\sqrt{1-4t(x-t)} = 1 - \tfrac{1}{2}4t(x-t) + \dots$$

Hence

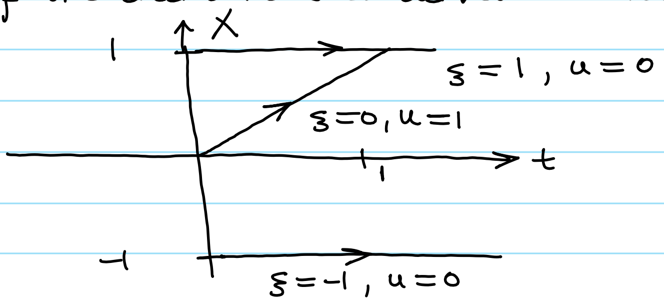
$$\frac{1 \pm \sqrt{1-4t(x-t)}}{2t} = \frac{1 \pm 1 \mp 2t(x-t)}{2t}$$

(*) Only by taking the $-$ do we get something finite as $t \to 0$. Hence

$$\xi = \frac{1 - \sqrt{1-4t(x-t)}}{2t} . \qquad (+)$$

(*) meaning: by taking the ~~up~~ lower of the two

Some of the characteristic curves are shown below:



We observe that the characteristics intersect for t large enough. The smallest t at which this occurs may be found as follows: (+) expresses ξ as a function of x and t, and we may write

$$u(x,t) = u_0(\xi(x,t))$$

If two characteristics intersect at (x,t) then $u_x(x,t)$ is infinite there [This is not immediately obvious, but you

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should think about it.] . Now, by the chain rule,

$$u_x(x,t) = u_0'(\xi(x,t))\,\xi_x$$

We can compute ξ_x directly from (+), or implicitly from (Q). The latter gives

$$1 = \xi_x - 2t\,\xi_x\,\xi$$

Hence

$$\xi_x = \frac{1}{1-2t\xi}$$

It follows that ξ_x — and therefore also u_x — becomes infinite when $t = \frac{1}{2\xi}$. The smallest positive value of t corresponds to $\xi = 1$.
Hence $t = 1/2$ is the first time at which two characteristics intersect. The soln method breaks down after that time.

Physically, this break down corresponds to the formation of a shock.