First-order systems We now turn our attention to (coupled) systems of the form

$$A \underline{u}_x + B \underline{u}_y = \underline{c} \qquad (S)$$

Here, $\underline{u}$ is a vector of n unknowns; they are functions of the independent variables x and y. A and B are $n \times n$ matrices that may depend on x, y and $\underline{u}$, whilst $\underline{c}$ is a vector of n components (that may also depend on x, y and $\underline{u}$).

Example The motion of a shallow layer of fluid is governed by the eqns

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}[uh] = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x}$$

Where h is the height of the layer, u is the velocity of the fluid, ~~and~~ and g is the acceleration due to gravity (a constant) We can recast these eqns in matrix form

$$\begin{pmatrix} h_t \\ u_t \end{pmatrix} + \begin{pmatrix} u & h \\ g & u \end{pmatrix}\begin{pmatrix} u_x \\ h_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So

$$A = I, \quad B = \begin{pmatrix} u & h \\ g & u \end{pmatrix} \quad \text{and} \quad \underline{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Characteristic curves for systems We study how the solution changes along the curve of eqn $y = Y(x)$. We have

$$\frac{d}{dx}\underline{u}(x, Y(x)) = \underline{u}_x(x, Y(x)) + Y'(x)\,\underline{u}_y(x, Y(x))$$

We can therefore eliminate the $\underline{u}_x$ term in (S):

$$A\frac{d}{dx}\underline{u}(x, Y(x)) + [B - Y'A]\,\underline{u}_y(x, Y(x)) = \underline{c} \qquad (*)$$

along $y = Y(x)$. In the case $n=1$, we could choose Y' so as to eliminate the $\underline{u}_y$ term from $(*)$.

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In the case $n>1$, this is not possible, but we can ask that
$$B - Y'A$$
be singular; equivalently, choose $Y(x)$ so that

$$|B - Y'A| = 0 \qquad (C)$$

Definition The curves of eqn $y = Y(x)$ such that (C) holds are called characteristics of (S).

Aside: The problem of finding $\lambda \in \mathbb{C}$ and $\underline{v} \neq \underline{0}$ s.t.
$$B\underline{v} = \lambda A \underline{v}$$
is called the generalised eigenvalue problem for (A, B).

The vector $\underline{v}$ is called a "right" eigenvector corresponding to the eigenvalue λ. There is also the concept of "left" eigenvector $\underline{w}$, which satisfies
$$\underline{w}^t B = \lambda \underline{w}^t A$$
The following are elementary facts of Linear Algebra: $\underline{w}$ is a left eigenvector of (A, B) iff it is a right eigenvector of (A^t, B^t) corresponding to the same eigenvalue.

Back to characteristic curves: an equivalent way of phrasing our defn of characteristic curve is to say that they are curves s.t.
$$Y' = \lambda$$
where λ is an eigenvalue of (A, B). For a system of n eqns, there may be as many as n distinct eigenvalues λ_j. Even in cases where some of the eigenvalues are repeated, it may still be possible to find n linearly independent eigenvectors.

Definition The system
$$A\underline{u}_x + B\underline{u}_y = \underline{c} \qquad (S)$$
is called hyperbolic if (1) the eigenvalues are all real and (2) there are n linearly independent eigenvectors.

Now, let λ be an eigenvalue of (A, B). Then there is at least one left-eigenvector $\underline{v}$. Multiply Eqn $(*)$ on p. 1 by $\underline{v}^t$ and suppose that $Y' = X$. Then, since

$$\underline{v}^t \left[B - \lambda A \right] = \underline{0}$$

we find

$$\underline{v}^t A \frac{d\underline{u}}{dx} = \underline{v}^t \underline{c} \qquad \text{along } Y' = \lambda$$

For a hyperbolic system, there are n linearly independent eigenvectors $\underline{v}_j$, $1 \le j \le n$, with corresponding eigenvalues λ_j and we can write

$$\underline{v}_j^t A \frac{d}{dx} \underline{u} = \underline{v}_j^t \underline{c} \qquad \text{along } Y' = \lambda_j \quad , \; 1 \le j \le n$$

This is the <u>characteristic form</u> of the system (S)

<u>Example</u> For the shallow fluid layer considered earlier

$$0 = |B - \lambda A| = \begin{vmatrix} u-\lambda & h \\ g & u-\lambda \end{vmatrix} = \lambda^2 - 2u\lambda + u^2 - gh$$

Hence $\lambda_\pm = u \pm \sqrt{gh}$ and the system is hyperbolic.
To work out the characteristic form, we need to find, for each eigenvalue, a left-eigenvector. Since $A = I$, these are left-eigenvectors of B or, equivalently, right-eigenvectors of B^t

$\underline{\lambda = \lambda_-}$: $$B^t - \lambda_- I = \begin{pmatrix} \sqrt{gh} & g \\ h & \sqrt{gh} \end{pmatrix} \to \begin{pmatrix} 1 & \sqrt{g/h} \\ 0 & 0 \end{pmatrix}$$

So $\underline{v}_- = \begin{pmatrix} -\sqrt{g} \\ \sqrt{h} \end{pmatrix}$.

$\underline{\lambda = \lambda_+}$: $$B^t - \lambda_+ I = \begin{pmatrix} -\sqrt{gh} & g \\ h & -\sqrt{gh} \end{pmatrix} \to \begin{pmatrix} 1 & -\sqrt{g/h} \\ 0 & 0 \end{pmatrix}$$

So $\underline{v}_+ = \begin{pmatrix} \sqrt{g} \\ \sqrt{h} \end{pmatrix}$.

The characteristic form is thus

(1) $$-\sqrt{g}\,\frac{dh}{dx} + \sqrt{h}\,\frac{du}{dx} = 0 \quad \text{along} \quad Y' = u - \sqrt{gh}$$

(2) $$\sqrt{g}\,\frac{dh}{dx} + \sqrt{h}\,\frac{du}{dx} = 0 \quad \text{along} \quad Y' = u + \sqrt{gh}$$

The characteristic form expresses that some quantities are conserved along characteristic curves; in the present case, (1) says

$$\frac{d}{dx}\left[u - 2\sqrt{gh} \right] = 0 \quad \text{along} \quad Y' = u - \sqrt{gh}$$

whilst (2) says

$$\frac{d}{dx}\left[u + 2\sqrt{gh} \right] = 0 \quad \text{along} \quad Y' = u + \sqrt{gh}$$

<u>Example</u> Consider the system

$$u_x + u_y + v_y = 0$$
$$v_x + u_y + 2v_y + w_y = 0$$
$$w_x - u_y + 2v_y = 0$$

Here $n=3$. In matrix form:

$$\begin{pmatrix} u_x \\ v_x \\ w_x \end{pmatrix} + \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -1 & 2 & 0 \end{pmatrix}}_{B} \begin{pmatrix} u_y \\ v_y \\ w_y \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\underline{0}}$$

So

$A = I$ and the eigenvalues are simply the eigenvalues of B:

$$0 = |B - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 2 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ -1 & -\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda - 3\right] = (1-\lambda)(\lambda-3)(\lambda+1).$$

$\underline{\lambda_1 = -1}$. We find $\underline{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\underline{\lambda_2 = 1}$. We find $\underline{v}_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

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$\lambda_3 = 3$. We find $v_3 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$.

So the characteristic form of the system is

$$\frac{d}{dx}[u - v + w] = 0 \quad \text{along} \quad y = -x + \text{const}$$

$$\frac{d}{dx}[-3u + v + w] = 0 \quad \text{along} \quad y = x + \text{const}$$

$$\frac{d}{dx}[u + 3v + w] = 0 \quad \text{along} \quad y = 3x + \text{const}.$$

Classification of second-order pdes Consider

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g \qquad (E)$$

Here, the coefficients are assumed constant (for the moment).

Remark We may without loss of generality suppose that $f = 0$. Indeed, if this were not the case, we could introduce a new unknown, say $\tilde{u}$, via

$$e^{rx}\tilde{u} = u$$

Then

$$u_x = r e^{rx}\tilde{u} + e^{rx}\tilde{u}_x, \quad u_y = e^{rx}\tilde{u}_y \quad \text{etc}$$

and, by choosing r judiciously, we can arrange so that the eqn for $\tilde{u}$ contains no term $\tilde{f}\tilde{u}$.

With this in mind, set $v = u_x$ and $w = u_y$. Then

$$a v_x + 2b v_y + c w_y + d v + e w = g \qquad (1)$$

To express (E) as a system, we require an additional eqn. Use

$$v_y - w_x = 0$$

This simply expresses the fact that

$$u_{xy} = u_{yx} \qquad (2)$$

We put (1) and (2) in matrix form:

$$\underbrace{\begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} v_x \\ w_x \end{pmatrix}}_{\underline{u}_x} + \underbrace{\begin{pmatrix} 1 & 0 \\ 2b & c \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} v_y \\ w_y \end{pmatrix}}_{\underline{u}_y} = \underbrace{\begin{pmatrix} 0 \\ g-dv-ew \end{pmatrix}}_{\underline{c}}$$

This is the case $n=2$ of the system we discussed last week. [with particular choices of A and B (and $\underline{c}$)].

What are the characteristics?

$$0 = |B-\lambda A| = \begin{vmatrix} 1 & \lambda \\ 2b-\lambda a & c \end{vmatrix} = a\lambda^2 - 2b\lambda + c = 0 \qquad (+)$$

We shall henceforth assume that $a \neq 0$. There is no great loss of generality in doing this. Indeed, if $a=0$ and $c=0$, the eqn is in a very simple form [We shall see shortly that it is then in the (hyperbolic) canonical form] On the other hand, if $a=0$ and $c \neq 0$, we can swap x and y to revert to the case $a \neq 0$. With this assumption, the eigenvalues are given by

$$\lambda_\pm = \frac{b \pm \sqrt{b^2-ac}}{a}$$

<u>Definition</u> The eqn (E) is said to be of

(1) <u>hyperbolic</u> type if $b^2 > ac$ [e.g. $u_{yy} = c^2 u_{xx}$]

(2) <u>parabolic</u> type if $b^2 = ac$ [e.g. $u_y = \sigma^2 u_{xx}$]

(3) <u>elliptic</u> type if $b^2 < ac$ [e.g. $u_{yy} + u_{xx} = 0$]

The wave, heat and Laplace eqns provide examples of each type. We now show that each type may be reduced to a simple <u>canonical</u> form. We begin by discussing how the eqn changes under a transformation of the independent variables

Set $u(x,y) = \mathcal{U}(\xi(x,y), \eta(x,y))$

Then

$$u_x = \mathcal{U}_\xi \xi_x + \mathcal{U}_\eta \eta_x \quad \text{and} \quad u_y = \mathcal{U}_\xi \xi_y + \mathcal{U}_\eta \eta_y$$

and so on for the second-order derivatives. The pde satisfied by $\mathcal{U}(\xi,\eta)$ is of the form

$$A\,\mathcal{U}_{\xi\xi} + 2B\,\mathcal{U}_{\xi\eta} + C\,\mathcal{U}_{\eta\eta} + \text{first-order terms} = g$$

where

$$A = a\xi_x^2 + 2b\,\xi_x\xi_y + c\,\xi_y^2$$

$$B = a\,\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\,\xi_y\eta_y$$

$$C = a\eta_x^2 + 2b\,\eta_x\eta_y + c\,\eta_y^2$$

It is an interesting exercise to comput $B^2 - AC$ ...
You should find that it is of the same sign as $b^2 - ac$.
Therefore, the type does not change under an arbitrary transformation of the independent variables.

We now discuss specific choices of ξ and η, according to the type of the eqn.

(1) The hyperbolic case: There are two real eigenvalues λ_- and λ_+. Integration of

$$Y' = \lambda_-$$

yields an implicit formula for Y in terms of x:

$$\xi(x, Y) = \text{const}$$

Likewise, integration of

$$Y' = \lambda_+$$

yields

$$\eta(x, Y) = \text{const}.$$

We choose ξ and η as new variables. It should be noted that, with this choice, we have

$$\xi_x + \lambda_- \xi_y = 0 \quad \text{and} \quad \eta_x + \lambda_+ \eta_y = 0$$

It follows that

$$A := a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = \xi_y^2\,[a\lambda_-^2 - 2b\lambda_- + c] = 0$$

and

$$C := a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = \eta_y^2\,[a\lambda_+^2 - 2b\lambda_+ + c] = 0$$

As for B, a straightforward calculation yields

$$B = -\frac{4}{a}\,\xi_y\eta_y\,(b^2 - ac) \neq 0$$

So we can write the eqn for u in the form

$$u_{\xi\eta} + E\,u_\xi + F\,u_\eta = G$$

This is the <u>canonical form</u> for the hyperbolic type.

(2) The parabolic case. In this case, there is just one eigenvalue. Integrating

$$Y' = \lambda \qquad [= b/a]$$

yields $\quad \xi(x, Y) = \text{const.}$

The other variable η is chose arbitrarily; all we ask is that ξ and η be independent, i.e.

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 .$$

Then $A = 0$ as before. Furthermore

$$\begin{aligned} B &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\,\xi_y\eta_y \\ &= \xi_y\,[-a\lambda\eta_x - b\lambda\eta_y + b\eta_x + c\eta_y] \\ &= 0 \qquad \text{since } \lambda = b/a \text{ and } b^2 = ac . \end{aligned}$$

Finally

since $b^2 = ac$

$$C := a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = a\left[\eta_x + \frac{b}{a}\eta_y\right]^2$$

$$= \frac{a}{\xi_y^2}\left(\xi_x\eta_y - \xi_y\eta_x\right)^2 \quad \text{since } b/a = -\xi_x/\xi_y$$

So $C \neq 0$ and we can write the eqn for $\mathcal{U}$ in the form

$$\mathcal{U}_{\eta\eta} + E\mathcal{U}_\xi + F\mathcal{U}_\eta = G$$

This is the canonical form for parabolic eqns.

(3) The elliptic case. In this case, $\lambda_\pm$ form a complex conjugate pair. We find that the variables ξ and η — defined as in the hyperbolic case —

$$Y' = \lambda_- \implies \xi(x, Y) = \text{const}$$
$$Y' = \lambda_+ \implies \eta(x, Y) = \text{const}$$

form a complex conjugate pair. So we can write

$$\xi = s + it \quad \text{and} \quad \eta = s - it$$

where s and t are REAL. It may then be shown that by setting

$$u(x,y) = \mathcal{U}(s(x,y), t(x,y))$$

we obtain

$$\mathcal{U}_{ss} + \mathcal{U}_{tt} + E\mathcal{U}_s + F\mathcal{U}_t = G$$

This is the canonical form for the elliptic type.

<u>Example</u> Put the wave eqn

$$u_{xx} = \mu^2 u_{yy}$$

in canonical form.

<u>Soln</u>: Here $a = 1$, $b = 0$, $c = -\mu^2$. Hence $\lambda_\pm = \pm\mu$

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We have

$$Y' = \lambda_- = -\mu \quad . \quad \text{Hence} \quad Y = -\mu x + \text{const}$$
$$Y' = \lambda_+ = \mu \quad . \quad \text{Hence} \quad Y = \mu x + \text{const}.$$

So

$$\xi(x,y) = y + \mu x \quad \text{and} \quad \eta(x,y) = y - \mu x$$

Put

$$u(x,y) = \mathcal{U}(y+\mu x, y-\mu x)$$

Then

$$u_x = \mu \mathcal{U}_\xi - \mu \mathcal{U}_\eta \quad , \quad u_y = \mathcal{U}_\xi + \mathcal{U}_\eta$$
$$u_{xx} = \mu^2 \mathcal{U}_{\xi\xi} - \mu^2 \mathcal{U}_{\xi\eta} - \mu^2 \mathcal{U}_{\eta\xi} + \mu^2 \mathcal{U}_{\eta\eta}$$
$$u_{yy} = \mathcal{U}_{\xi\xi} + \mathcal{U}_{\xi\eta} + \mathcal{U}_{\eta\xi} + \mathcal{U}_{\eta\eta}$$
$$u_{xy} = \mu \mathcal{U}_{\xi\xi} + \mu \mathcal{U}_{\xi\eta} - \mu \mathcal{U}_{\xi\eta} - \mu \mathcal{U}_{\eta\eta} \qquad [\text{not needed!}]$$

We deduce

$$\begin{aligned} 0 &= u_{xx} - \mu^2 u_{yy} \\ &= [\mu^2 - \mu^2] \mathcal{U}_{\xi\xi} + [-2\mu^2 - 2\mu^2] \mathcal{U}_{\xi\eta} + [\mu^2 - \mu^2] \mathcal{U}_{\eta\eta} \end{aligned}$$

So the canonical form is

$$\mathcal{U}_{\xi\eta} = 0$$

This can be integrated:

$$\mathcal{U}(\xi,\eta) = f(\xi) + g(\eta)$$

Hence

$$u(x,y) = f(y+\mu x) + g(y - \mu x)$$

and we recover a well-known result!

<u>Example</u> $3u_{xx} + 10u_{xy} + 3u_{yy} + 4u_x + 5u_y + u = g(x,y)$

Here $a=3$, $b=5$ and $c=3$. So $b^2 - ac = 16$ and the eqn is hyperbolic. We have

$$\lambda_\pm = \frac{5}{3} \pm \frac{4}{3}$$

Hence

$$Y' = \lambda_- = \frac{1}{3} \quad \rightarrow \quad Y = \frac{x}{3} + \text{const} \quad \text{and so} \quad \xi(x,y) = 3y - x$$
$$Y' = \lambda_+ = 3 \quad \rightarrow \quad Y = 3x + \text{const} \quad \text{and so} \quad \eta(x,y) = y - 3x$$

Note: ξ and η are not uniquely defined by the prescription

$Y' = \lambda_-$ [for ξ] and $Y' = \lambda_+$ [for η].

There is an arbitrary constant factor, which I chose here to avoid fractions.

Put $u(x,y) = \mathcal{U}(3y-x,\, y-3x)$. Then $x = \frac{\xi - 3\eta}{8}$

$y = \frac{3\xi - \eta}{8}$

$$u_x = -\mathcal{U}_\xi - 3\mathcal{U}_\eta, \quad u_y = 3\mathcal{U}_\xi + \mathcal{U}_\eta$$

$$u_{xx} = \mathcal{U}_{\xi\xi} + 3\mathcal{U}_{\xi\eta} + 3\mathcal{U}_{\eta\xi} + 9\mathcal{U}_{\eta\eta}$$

$$u_{xy} = -3\mathcal{U}_{\xi\xi} - 3\mathcal{U}_{\xi\eta} - \mathcal{U}_{\eta\xi} - 3\mathcal{U}_{\eta\eta}$$

$$u_{yy} = 9\mathcal{U}_{\xi\xi} + 3\mathcal{U}_{\xi\eta} + 3\mathcal{U}_{\eta\xi} + \mathcal{U}_{\eta\eta}$$

Hence

$$g\left(\frac{\xi-3\eta}{8}, \frac{3\xi-\eta}{8}\right) = \mathcal{U} + [-12+5]\,\mathcal{U}_\eta + [-12+15]\,\mathcal{U}_\xi$$

$$+ [27-30+3]\,\mathcal{U}_{\eta\eta} + [18-40+18]\,\mathcal{U}_{\xi\eta} + [3-30+27]\,\mathcal{U}_{\xi\xi}$$

So the canonical form is

$$-\frac{1}{4}\, g\left(\frac{\xi-3\eta}{8}, \frac{3\xi-\eta}{8}\right) = -\frac{1}{4}\mathcal{U} + \frac{7}{4}\mathcal{U}_\eta - \frac{3}{4}\mathcal{U}_\xi + \cancel{4}\,\mathcal{U}_{\xi\eta} \qquad (+)$$

Remark This eqn contains the term u on the ~~right~~ left-hand side. To put the eqn in the general form discussed earlier — where $f=0$ — we should really remove this term by making first the substitution $u = e^{\gamma x}\tilde{u}$. Try it, and reduce the eqn for $\tilde{u}$ to canonical form. Compare with (+).

Example $u_{xx} - 2u_{xy} + 5u_{yy} = 0$ [$a=1, b=-1, c=5$]

This eqn is elliptic! We have $\lambda_\pm = -1 \pm 2i$ and so

$$\xi = y + (1+2i)x \quad \text{and} \quad \eta = y + (1-2i)x$$

Putting $\xi = s+it$ and $\eta = s - it$ yields $s = x+y$ and $t = 2x$.

Set $u(x,y) = \mathcal{U}(x+y, 2x)$. Then

$$u_x = \mathcal{U}_s + 2\mathcal{U}_t, \quad u_y = \mathcal{U}_s$$

$$u_{xx} = \mathcal{U}_{ss} + 2\mathcal{U}_{st} + 4\mathcal{U}_{tt}, \quad u_{xy} = \mathcal{U}_{ss} + 2\mathcal{U}_{st}, \quad u_{yy} = \mathcal{U}_{ss}$$

We find the canonical form

$$[1-2+5]\,\mathcal{U}_{ss} + [4-4]\,\mathcal{U}_{st} + 4\,\mathcal{U}_{tt} = 0$$