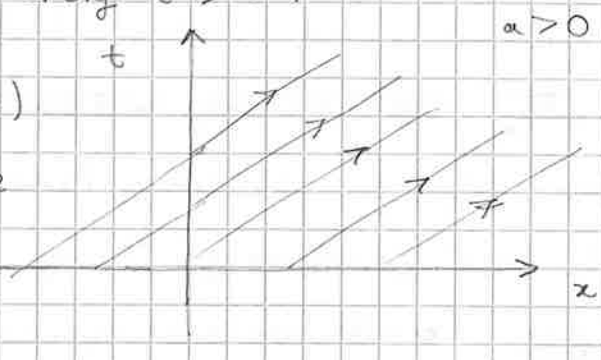
Hyperbolic problems Consider a system of eqns of the form
$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = f \quad , \quad (x,t) \in D$$
where $f : D \to \mathbb{R}^d$ and $A : D \to$ real $d\times d$ matrices are known, and $u : D \to \mathbb{R}^d$ is the unknown. The system is said to be hyperbolic if, for every $(x,t) \in D$, the matrix A has real eigenvalues and is diagonalisable.

Example The simplest example corresponds to $d = 1$ and
$$u_t + a u_x = 0 \quad , \quad x \in \mathbb{R} , \ t > 0 .$$
where a is a constant. The pure initial-value problem for this eqn may be solved by the method of characteristics. For this simple example, the characteristic curves $x = X(t)$ satisfy
$$\frac{dX}{dt} = a \quad , \text{ i.e. } \quad X(t) = x_0 + at \quad , \quad x_0 \in \mathbb{R} .$$
The evolution of the solution along such curves is governed by the equation
$$\frac{d}{dt}\, u(X(t),t) = u_t(X(t),t) + u_x(X(t),t)\frac{dX}{dt}$$
$$= u_t(X(t),t) + a\, u_x(X(t),t) = 0$$
and so
$$u(X(t),t) = u(x_0,0) \quad \text{for every } t \geq 0 .$$
We deduce
$$u(x,t) = u(x-at, 0)$$



More generally, let $e_1, \dots, e_d$ denote the eigenvectors of A corresponding to the eigenvalues $\lambda_1, \dots, \lambda_d$.

Since the system is, by assumption, diagonalisable, we can write
$$u = \alpha_1 e_1 + \dots + \alpha_d e_d$$
for some real-valued functions $\alpha_1, \dots, \alpha_d$. In that basis, the system takes the form
$$\frac{\partial}{\partial t}\alpha_j + \lambda_j \frac{\partial \alpha_j}{\partial x} = \varphi_j(x,t;\alpha_1,\dots,\alpha_d)$$
In particular, if A is a constant matrix and f the zero-vector, we have d decoupled eqns of the form in the example.

Example The wave eqn is

$$e_{tt} = c^2 e_{xx}$$

is hyperbolic. To see this, put

$$u = \begin{pmatrix} e_t \\ e_x \end{pmatrix}$$

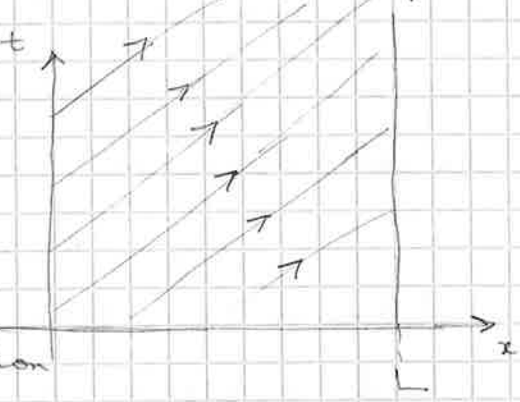
Then

$$\frac{\partial u}{\partial t} = \begin{pmatrix} e_{tt} \\ e_{xt} \end{pmatrix} = \begin{pmatrix} c^2 e_{xx} \\ e_{tx} \end{pmatrix} = \begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix} \frac{\partial u}{\partial x}$$

So $d = 2$ and $A = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}$. The eigenvalues are $\lambda = \pm c$.

Boundary conditions. Well-posedness. Let $a > 0$ and consider the problem

$$u_t + a u_x = 0, \quad (x,t) \in D$$



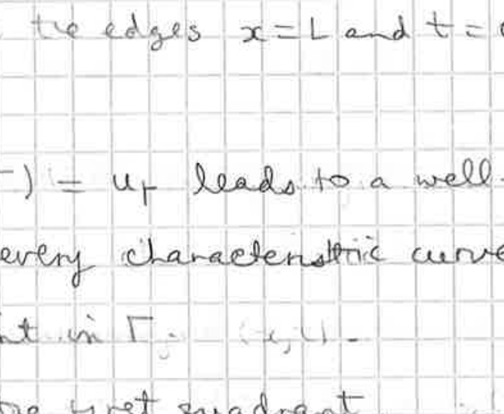
where D is a vertical strip. What boundary condition can one use to specify a unique soln?

It is clear that we can use a boundary condition that specifies the value of the solution along the edges $x = 0$ and $t = 0$.

On the other hand, specifying the value at the edges $x = L$ and $t = 0$ would lead to an ill-posed problem. (Why?)

Generally, the boundary condition $u(\Gamma) = u_\Gamma$ leads to a well-posed problem only if, for every $(x,t) \in D$, every characteristic curve through (x,t) first passes through a point in Γ.

Example Consider the wave equation in the first quadrant.



The characteristics that go through the point (x,t) are of the form

$x = x_0 + ct$ and $x = x_0 - ct$.

We see that a well-posed problem is obtained by specifying $u(0,t)$, $u(L,t)$ and $u(x,0)$.

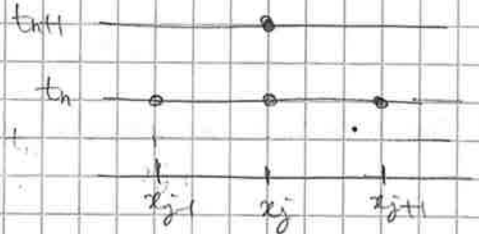
Some difference schemes The construction and the analysis of difference schemes follow essentially the same principles as in the parabolic case

There is, however, one additional tool that is useful in studying stability, called the CFL condition, which we proceed to illustrate by using specific schemes for the equation

$$u_t + a u_x = 0, \quad x \in \mathbb{R},\ t > 0$$

<u>Example</u> One-sided difference scheme

$$u_j^{n+1} = u_j^n - \frac{ak}{h}\left(u_{j+1}^n - u_j^n \right)$$



The CFL condition for this scheme is as follows: the scheme is stable only if the characteristic curve through (x_j, t_{n+1}) crosses the line $t = t_n$ between x_j and x_{j+1}.

Now, the characteristic through (x_j, t_{n+1}) is the curve of eqn

$$x = x_j - a(t_{n+1} - t)$$

So the CFL condition is

$$x_j < x_j - ak < x_j + h$$

That is

$$0 < -ak < h$$

This condition is always violated if $a > 0$. For $a < 0$, it places a restriction on the ratio k/h.

Indeed, if we put $u_j^n = g^n e^{i\xi x_j}$, $\xi h \in [0, 2\pi)$ we see that

$$g = 1 - \frac{ak}{h}\left(e^{i\xi h} - 1 \right)$$

$$= 1 - \frac{ak}{h}(\cos \xi h - 1) - i\frac{ak}{h}\sin \xi h$$

Hence

$$|g|^2 = 1 - \frac{2ak}{h}(\cos \xi h - 1) + \frac{a^2k^2}{h^2}(2 - 2\cos \xi h)$$

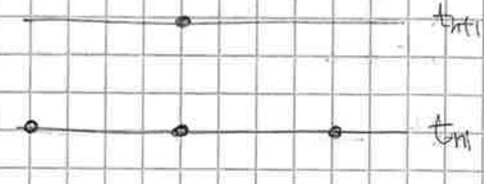
$$> 1$$

if $\quad \frac{a^2k^2}{h^2}(1 - \cos \xi h) > \frac{ak}{h}(\cos \xi h - 1)$

<u>Example</u> The "Lax-Wendroff" scheme is

$$u_j^{n+1} = u_j^n - \frac{1}{2}\mu\left(u_{j+1}^n - u_{j-1}^n \right) + \frac{1}{2}\mu^2\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right)$$

with $\mu = -ak/h$



It is easy to see that this scheme is second-order accurate in space and time. The CFL condition is then

$$x_j - h < x_j - ak < x_j + h, \quad \text{i.e.} \quad k < \frac{h}{|a|}$$

On the other hand, von Neumann analysis yields

$$g = 1 - \frac{\mu}{2}\left(e^{i\xi h} - e^{-i\xi h}\right) + \frac{1}{2}\mu^2\left(e^{i\xi h} - 2 + e^{-i\xi h}\right)$$

$$= 1 - i\mu \sin \xi h - 2\mu^2 \sin^2 \frac{\xi h}{2}$$

Hence

$$|g|^2 = 1 - 4\mu^2 \sin^2 \frac{\xi h}{2} + 4\mu^2 \sin^2 \frac{\xi h}{2} \cos^2 \frac{\xi h}{2} + 4\mu^4 \sin^4 \frac{\xi h}{2}$$

$$= 1 - 4\mu^2 \sin^4 \frac{\xi h}{2} + 4\mu^4 \sin^4 \frac{\xi h}{2}$$

and the scheme is stable iff $\mu^2 \leq 1$.