Numerical dispersion relations. Every time-dependent, scalar, linear PDE with constant coefficients admits solns of the form

$$u(x,t) = e^{i[\xi x + \omega t]}, \quad \xi \in \mathbb{R},\ \omega \in \mathbb{C}$$

We may view the frequency ω as a function of the wave number ξ, i.e.

$$\omega = \omega(\xi) \qquad (D)$$

This is called the dispersion relation associated with the pde.

Defn Consider the dispersion relation (D). If ω is real for ξ real, we define the phase speed c and the group speed c_g by

$$c = -\frac{\omega}{\xi}, \quad c_g = -\frac{d\omega}{d\xi}$$

Obviously, c quantifies how fast the plane wave moves through the spatial domain. The interpretation of the group speed is not so obvious; it determines how a soln made up of plane waves decays for large time.

Examples

$u_t + a u_x = 0$. $\omega(\xi) = -a\xi$, $c = a = c_g$.

$u_{tt} - a^2 u_{xx} = 0$. $\omega_\pm = \pm a\xi$, $c_\pm = \mp a = c_{g\pm}$

$u_t = \frac{\sigma^2}{2} u_{xx}$. $i\omega = -\frac{\sigma^2}{2}\xi^2$, $\omega = i\frac{\sigma^2}{2}\xi^2$

$u_t = i u_{xx}$. $\omega = -\xi^2$, $c = \xi$, $c_g = 2\xi$

Defn The plane wave is said to be dissipative if $\operatorname{Im}\omega(\xi) > 0$ and dispersive if $c(\xi) \neq c_g(\xi)$.

Thus, the plane wave solns of the heat eqn are dissipative (their amplitude is decaying with time) and those of the Schrödinger eqn are dispersive (a soln initially made up of a superposition of plane waves will "unravel" as time evolves.)

When these pdes are discretised, the numerical scheme will also support plane wave solns of the form

$$u_j^n = e^{i[\xi x_j + \omega t_n]}, \quad \xi h \in [0, 2\pi]$$

The relationship between ω and ξ is then called the numerical dispersion relation. It is interesting to study how well it reproduces the true dispersion relation.

Example Consider the Forward Euler scheme

$$u_j^{n+1} = u_j^n + \frac{\mu}{2}\left[u_{j+1}^n - u_{j-1}^n \right] , \quad \mu = -\frac{ak}{h}$$

Put $u_j^n = e^{i[\xi x_j + \omega t_n]}$, $\xi h \in [0, 2\pi]$

Then, after simplification,

$$e^{i\omega k} = 1 + \frac{\mu}{2}\left[e^{i\xi h} - e^{-i\xi h} \right] = 1 + i\mu \sin \xi h \qquad (N)$$

and we obtain the numerical dispersion relation

$$\omega = \frac{1}{ik} \ln\left(1 + i\mu \sin \xi h \right) \qquad (+)$$

Consider the limit $k \to 0$ with μ and ξ fixed. Using $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$ we find

$$\omega = \frac{1}{ik}\left\{ i\mu \sin \xi h + \frac{\mu^2}{2} \sin^2 \xi h + i\frac{\mu^3}{3} \sin^3 \xi h \ldots \right\}$$

$$= -a\xi \frac{\sin \xi h}{\xi h} - i\frac{k}{2} a^2 \xi^2 \frac{\sin^2 \xi h}{\xi^2 h^2} - \frac{k^2}{3} a^3 \xi^3 \frac{\sin^3 \xi h}{\xi^3 h^3} + \ldots$$

Put $\omega = \omega_r + i\omega_i$. Then

$$\omega_r = -a\xi \frac{\sin \xi h}{\xi h} - \frac{k^2}{3} a^3 \xi^3 \frac{\sin^3 \xi h}{\xi^3 h^3} + \ldots$$

$$= -a\xi\left[1 - \frac{\xi^2 h^2}{6} + \ldots \right] - \frac{k^2}{3} a^3 \xi^3 \left[1 - \ldots \right]$$

$$= \underbrace{-a\xi}_{\text{true dispersion relation}} + \underbrace{\frac{a\xi^3}{6} h^2 \left(1 - 2\mu^2\right) + \ldots}_{\text{numerical dispersion error}}$$

and

$$\omega_i = \underbrace{-k\frac{a^2\xi^2}{2}}_{\text{numerical (anti) dissipation error}} + \ldots$$

We see that, for ξh small, (+) approximates the true dispersion well. For ξh not so small, however, the discretisation introduces both artificial dispersion and an anti-dissipative term; the amplitude of the plane wave discrete soln will tend to grow with n. Other examples may be found in the problem sheet.