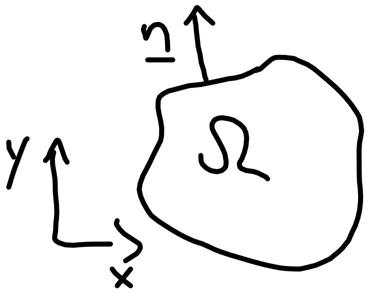
4 Finite Difference Methods for Elliptic Equations

Most important example: Poisson's equation

$$-\Delta u = -\nabla^2 u = -(u_{xx} + u_{yy}) = f(x,y)$$



$(x,y) \in \Omega$

prescribe $u|_{\partial\Omega} = g(x,y)$

(Dirichlet problem)

or prescribe $\underline{n} \cdot \nabla u|_{\partial\Omega} = g$

v. Neumann

§ 4.1 1D example: $-u_{xx} = f(x)$

$x \in [0,1]$ $f \in C^0$

$u(0) = u(1) = 0$

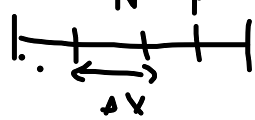
Properties

a) $u(x)$ ex. and unique

b) $\|u\|_\infty \le \frac{1}{8} \|f\|_\infty = \frac{1}{8} \max\limits_{(0,1)} f$

Define grid $x_j = j\Delta x$

$N=4$ $j = 0 \ldots N$



$\Delta x = \frac{1}{N}$

2nd order finite diff.

$$-\left(\frac{u_{j-1} - 2u_j + u_{j+1}}{(\Delta x)^2}\right) = f(x_j) \qquad j = 1, \ldots N-1$$

$$u_0 = u_N = 0$$

Matrix equation

$$\frac{1}{\Delta x^2}\begin{bmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix} \underline{u} = \underline{f}$$

$\overset{!!}{\underline{\underline{A}}}$ $(N-1)\times(N-1)$ matrix

[eigenvalues:

$$\lambda_n = 4\sin^2\left(\frac{\pi n}{2(N+1)}\right)$$

$n = 1, \ldots, N-1$

$\Rightarrow A$ pos. def!

Hence $\underline{v}^T \underline{\underline{A}}\, \underline{v} > 0$, $\underline{v} \neq 0$

$\Rightarrow A^{-1}$ ex. $\rightsquigarrow$ unique sol $\boxed{\underline{u} = \underline{\underline{A}}^{-1} \underline{f}}$ a)

Want convergence for $\Delta x \to 0$

$|u(x_j) - u_j| \to 0$

Facts $A^{-1} = \{\alpha_{ij}\}$

$\alpha_{ij} \geq 0$, $0 < \sum_{j=1}^{N} \alpha_{ij} \leq \frac{1}{8}$

Now $u_i = \alpha_{ij} f_j$

$\max u = \max(\alpha_{ij} f_j) \leq \max \sum_j \alpha_{ij} \|f\|_\infty$

$\|U\|_\infty \leq \frac{1}{8} \|f\|_\infty$ b)

Truncation error:

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{(\Delta x)^2} = \underbrace{u''(x_j)}_{\text{exact}} + \frac{(\Delta x)^2}{12} u^{iv}(\xi_j)$$

$\xi_j \in (x_{j-1}, x_{j+1})$

Let $\varepsilon_j := u(x_j) - u_j$

$$\frac{\varepsilon_{j+1} - 2\varepsilon_j + \varepsilon_{j-1}}{(\Delta x)^2} = \frac{(\Delta x)^2}{12} u^{iv}(\xi_j)$$

Or : $\underline{\underline{A}}\,\underline{\varepsilon} = \frac{(\Delta x)^2}{12} u^{IV}$, $\|\varepsilon\|_\infty \le \frac{1}{8}\frac{(\Delta x)^2}{12}\|u^{IV}\|_\infty$

assured 2nd order accuracy provided u^{IV} bounded

General case: $Lu = f$, difference scheme

Difference scheme $L u^{(N)} = f^{(N)}$

<u>consistent</u> $\left(L_N u(x_j) - f^{(N)}\right)_j \to 0$ f.a. j
$\Delta x \to 0$

Truncation error: $L_N u = f^{(N)} + \tau$

$\leadsto L_N \varepsilon = \tau$ since $L_N u^{(N)} = f^{(N)}$

$\leadsto \boxed{\varepsilon = L_N^{-1} \tau}$

Scheme is stable if $\|L^{-1}\|_\infty \le C$ ← independent of Δx

$$\|M\|_\infty = \sup_{v \in \mathbb{R}^{N-1}} \frac{\|Mv\|_\infty}{\|v\|_\infty}$$

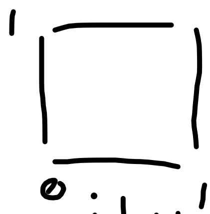
just as in time-dependent problems:

Consistency + stability ⇒ convergence

$\|\tau\|_\infty \to 0$ as $\Delta x \to 0$ $\quad \|L_N^{-1}\| \le C \quad$ $\|E\|_\infty \to 0$ as $\Delta x \to 0$

§ 4.2 extend to higher dimensions

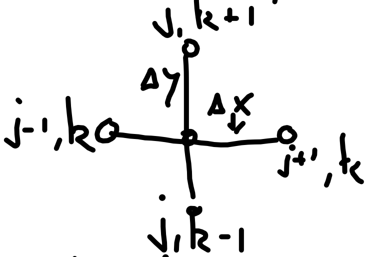
Poisson: $-(u_{xx}+u_{yy}) = f(x,y)$ $\quad x,y \in [0,1]$



2nd order finite diff

$$-\left(\frac{u_{j-1,k}-2u_{j,k}+u_{j+1,k}}{(\Delta x)^2}\right)$$

$$-\left(\frac{u_{j,k-1}-2u_{j,k}+u_{j,k+1}}{(\Delta y)^2}\right) = f(x_j, y_k)$$



$x_j = j\Delta x$

$y_k = k\Delta y$

$$\rightsquigarrow \underline{\underline{A}}\,\underline{u} = \underline{f}$$

$$\underline{\underline{A}} = \begin{bmatrix} A_x + 2I_y & -I_y & & 0 \\ -I_y & A_x + 2I_y & \ddots & \\ & \ddots & \ddots & -I_y \\ 0 & & -I_y & A_x + 2I_y \end{bmatrix}$$

$(N-1)(M-1) \times (N-1)(M-1)$

Define

$$A_x = \frac{1}{(\Delta x)^2} \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}$$

$$I_y = \frac{1}{(\Delta y)^2} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$(N-1) \times (N-1)$ matrices

block-tridiagonal: sparse!
banded structure, bandwidth $2(N-1)+1$

$\underline{\underline{A}}$ is positive definite: $\underline{u} = \underline{\underline{A}}^{-1} \underline{f}$ ex.
+ unique

can be shown: $\|A^{-1}\|_\infty \leq \frac{1}{8}$

$$\|\mathcal{E}\|_\infty \leq \frac{1}{96}\left[(\Delta x)^2 \max_{\Omega} |u_{xxxx}| + (\Delta y)^2 \max_{\Omega} u_{yyyy}\right]$$

Remaining task: find A^{-1} typically much too expensive for elimination technique: N^3 operation for $N \times N$ matrix

Instead: use iterative schemes, which only require multiplication $\rightsquigarrow$ use sparse nature of matrix.

§ 4.3 Gauss-Seidel

$$\begin{bmatrix} \ddots & U \\ L & \ddots \end{bmatrix} \leftarrow D$$

Write $A = \underset{\text{lower}}{L} + \underset{\text{diag.}}{D} + \underset{\text{upper}}{U}$

Iteration: $\boxed{(L+D)\,u^{n+1} = -U u^n + f}$ solution if $u^n = u^{n+1}$

componentwise:

$$u_i^{n+1} = \Big[f_i - \underbrace{\sum_{j=1}^{i-1} a_{ij}\, \overset{\text{found earlier}}{u_j^{n+1}}}_{\text{lower}} - \underbrace{\sum_{j=i+1}^{n} a_{ij}\, u_j^n}_{\text{upper}} \Big] \Big/ a_{ii}$$

§ Conjugate gradient for $\underline{A}\underline{x} = \underline{b}$

see [7] p. 293 Trefethen

Assume $\underline{\underline{A}}$ real symmetric + positive definite

$\rightsquigarrow$ Can define norm $\|x\|_A = \sqrt{x^T A x}$ on $\mathbb{R}^n$

$A: N \times N$

Let $\underline{x}^* = \underline{\underline{A}}^{-1} b$ exact,

$\underline{\varepsilon}_n = \underline{x}^* - \underline{x}_n$ error

minimization

$\phi = \frac{1}{2} \underline{x}^T \underline{\underline{A}} x - \underline{x}^T b$

$\underline{\nabla} \phi = \underline{\underline{A}} x - \underline{b} = 0$

Idea: Krylov space

$K_n = \langle b, Ab, A^2 b, \dots A^{n-1} b \rangle$

Method generates unique sequence $x_n \in K_n$

which minimizes $\|\varepsilon_n\|_A$

<u>Algorithm:</u> $\underline{x}_0 = 0;\ r_0 = b,\ \underline{P}_0 = \underline{r}_0$

for $n = 1, 2, 3, \dots$ $\quad \alpha_n = \dfrac{r_{n-1}^T r_{n-1}}{P_{n-1}^T A P_{n-1}}$ (step length)

$x_n = x_{n-1} + \alpha_n P_{n-1}$ (new it)

$r_n = r_{n-1} - \alpha_n A P_{n-1}$

$\beta_n = \dfrac{r_n^T r_n}{r_{n-1}^T r_{n-1}}$

$P_n = r_n + \beta_n P_{n-1}$ (update direction)

Only need 1 matrix comp $A P_{n-1}$!

Key facts: $K_n = \langle p_0, \dots p_{n-1} \rangle =$

(1) $\langle r_0, \dots, r_{n-1} \rangle = \langle x_1, \dots, x_n \rangle$

(2) $r_n^T r_j = 0 \quad j < n \qquad p_n^T A p_j = 0$

$\|\varepsilon_n\|_A$ minimal over K_n

$$\varepsilon_n + \delta\varepsilon_n = \varepsilon_n - \delta x_n$$

$$\|\varepsilon_n + \delta\varepsilon_n\|_A^2 = \|\varepsilon_n\|_A^2 - 2\underbrace{\varepsilon_n^T A}_{=[b - Ax_n]^T = r_n^T} \delta x_n + \|\delta x_n\|_A^2$$

$$= \|\varepsilon_n\|_A^2 - 2\cancel{r_n^T \delta x_n}^{\,0} + \underbrace{\|\delta x_n\|_A^2}_{>0}$$

$$\boxed{\|\varepsilon_n\|_A \geq \|\varepsilon_{n+1}\|_A}$$

can't get worse!

Algorithm must converge : $x^* \in K_{N}$

Either $K_N = \mathbb{R}^N$

or $K_N \subset \mathbb{R}^N$, so $A^{N-1} b \in \langle b, Ab \ldots A^{N-2} b \rangle = K_{N-1}$

or $A^{N-2} b \in \langle A^{-1} b, b, \ldots A^{N-3} b \rangle$

or $A^{-1} b \in \langle b, \ldots, A^{N-2} b \rangle$

$\leadsto x^* \in K_{N-1} \subseteq K_N$

Trefethen: $\kappa = \frac{\lambda_{max}}{\lambda_{min}}$

$$\frac{\|\varepsilon_n\|_A}{\|\varepsilon_0\|_A} \leq 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^n$$