§ Fourier spectral methods

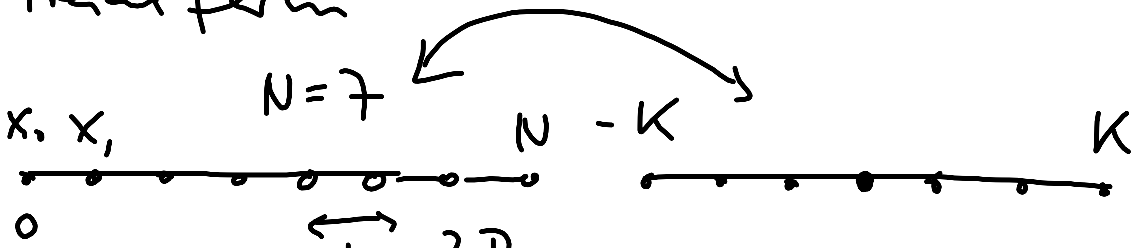
Why? ▷ easy to do derivatives $u \to \frac{du}{dx}$

▷ very accurate $\hat{u}_k \to k\,\hat{u}_k$ you

▷ fast

– mostly for periodic or unbounded domains

On a lattice, use discrete Fourier transform



$h = \frac{2\pi}{N}$ $\qquad N = 2K+1$

$x_i = \frac{2\pi i}{N}$

periodic: $u(x_0) = u(x_N)$

$$u_l = u(x_l) = \sum_{k=-K}^{K} \hat{u}_k e^{ikx_l}$$

$$\hat{u}_k = \frac{1}{N}\sum_{l=1}^{N} u_l e^{-ikx_l}$$

u real $\rightsquigarrow$ $\hat{u}_k \in \mathbb{C}$ $\quad \hat{u}_{-k} = \hat{u}_k^*$

Then $\frac{\partial u}{\partial x}(x) = \sum_{k=-K}^{K} ik\hat{u}_k e^{ikx}$

as mapping: $\hat{u}_k = F_k\{u_\ell\}$

$u_\ell = F_\ell^{-1}\{\hat{u}_k\}$

$\frac{2\pi j}{N}$

Fourier theorem: $u(x_j) = \sum_{k=-K}^{K} e^{ikx_j} \frac{1}{N}\sum_{\ell=1}^{N} u_\ell e^{-ikx_\ell}$

$= \frac{1}{N}\sum_{\ell=1}^{N} u_\ell \sum_{k=-K}^{K} e^{ik(j-\ell)\frac{2\pi}{N}} =$

$e^{-ik(j-\ell)\frac{2\pi}{N}}\left[1 + e^{i(j-\ell)\frac{2\pi}{N}} + \cdots + e^{2iK(j-\ell)\frac{2\pi}{N}}\right]$

$$\ldots = \begin{cases} e^{-ik(j-l)\frac{2\pi}{N}} \left(\dfrac{1 - e^{i\frac{(2k+1)}{N}(j-l)\frac{2\pi}{N}}}{1 - e^{i(j-l)\frac{2\pi}{N}}} \right) & j \neq l \\ 2k-1 = N & \text{if } j = l \end{cases}$$

$$= N\,\delta_{jl}\,, \quad \text{so} \quad u_j = \frac{1}{N}\sum_{l=1}^{N} u_l\, N\, \delta_{jl} = u_j$$

discrete k-values : periodic

bounded k-val : discrete spatial grid

3 different approaches:

A work in physical space

If $u(x,t) = \sum_{k=-K}^{K} \hat{u}_k(t) e^{ikx}$

$$u^{(p)}(x_\ell) = \sum_{-K}^{K} (ik)^p \hat{u}_k e^{ikx_\ell}$$

$$= \sum_{-K}^{K} (ik)^p e^{ikx_\ell} \left(\frac{1}{N} \sum_q u_q e^{-ikx_q} \right) =$$

$$= \sum_{q=1}^{N} \Bigg\{ \underbrace{\frac{1}{N} \sum_{-K}^{K} (ik)^p e^{ik(x_\ell - x_q)}}_{D^{(p)}_{\ell q}\text{: differentiation matrix.}} \Bigg\} u_q$$

$$l \neq q \quad D_{lq}^{(p)} := \frac{1}{N} \sum_{-K}^{K} \frac{d^p}{d\xi^p} e^{ik\xi} \quad , \quad \xi = x_l - x_q$$

$$= \frac{1}{N} \frac{d^p}{d\xi^p} \sum_{-K}^{K} e^{ik\xi} \quad = \frac{1}{N} \frac{d^p}{d\xi^p} \left(\frac{e^{i(K+1)\xi} - e^{-iK\xi}}{e^{i\xi} - 1} \right)$$

$$= \frac{1}{N} \frac{d^p}{d\xi^p} \left. \frac{\sin \frac{N\xi}{2}}{\sin \xi/2} \right|_{\xi = x_l - x_q}$$

$$l = q : \quad D_{ll}^{(p)} = \frac{1}{N} \sum_{-K}^{K} (ik)^p .$$

Ex. $u_t + c(x) u_x = 0 \quad x \in [0, 2\pi]$

$$\rightsquigarrow \frac{\partial \underline{u}}{\partial t} + c(x) \underline{\underline{D}}^{(1)} \underline{u} = 0 \qquad \underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}$$

B spectral space

1. Galerkin projection approach

$$\frac{1}{N} \sum_{l=1}^{N} (u_t + c(x) u_x) e^{-ikx_l} = 1$$

$$u(x,t) = \sum_{k'=-K}^{K} \hat{u}_{k'} e^{ik'x}$$

$$\Rightarrow \frac{\partial \hat{u}_k}{\partial t} + \sum_{k'=-K}^{K} \underbrace{ik' \left(\frac{1}{N} e^{i(k-k')x} \ldots \right)}_{M_{kk'}} \hat{u}_{k'} = 0$$

$$\text{so } \frac{\partial \hat{\underline{u}}}{\partial t} + \underline{\underline{M}}\, \hat{\underline{u}} = 0$$

diagonal if $c = \text{constant}$

$$u = \sum_{-K}^{K} \hat{u}_k e^{ik(x-ct)} \quad \leadsto \quad \frac{\partial \hat{u}_k}{\partial t} + ick\hat{u}_k = 0$$

$$\hat{u}_k = \hat{u}_k(0)\, e^{-ickt}$$

$$\leadsto \boxed{u = u(x-ct)}$$

2 Collocation approach

$u(x) = \sum_{-K}^{K} \hat{u}_k e^{ikx}$ plug into equation

$$\frac{\partial}{\partial t} \sum_{-K}^{K} \hat{u}_k e^{ikx_\ell} + c(x_\ell) \sum_{-K}^{K} ik \hat{u}_k e^{ikx_\ell}, \quad \ell = 1, \dots N$$

structure: $\underline{\underline{A}} \frac{\partial \hat{\underline{u}}}{\partial t} + \underline{\underline{B}}\, \hat{\underline{u}} = 0$, $\hat{u}$: sol. vector in k-space

either step this directly or

$$\boxed{\frac{\partial \hat{\underline{u}}}{\partial t} + \underline{\underline{A}}^{-1} \underline{\underline{B}}\, \hat{\underline{u}} = 0}$$

<u>spectral accuracy</u>: the smoother a fct, the more accurate a Fourier method becomes

Let $u(x)$ be a 2π-per function

Fourier coeff : $\hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u(x) e^{-ikx} dx$

$k = 0, \pm 1, \pm 2 \ldots$

Discr. F-transf. is a Projection

$$P_N u(x) = \sum_{k=-K}^{K} \hat{u}_k e^{-ikx}$$

Can show for continuous functions

$\| u - P_N u \|_\infty \xrightarrow[N\to\infty]{} 0$; easier in L_2 norm

$$\|u\|^2 = \int_0^{2\pi} |u(x)|^2 \, dx \,, \text{ so}$$

$$\|u\|^2 = \int_0^{2\pi} \left(\sum_{k_1=-\infty}^{\infty} \hat{u}_{k_1} e^{ik_1 x} \right) \left(\sum_{k_2=-\infty}^{\infty} \hat{u}^*_{k_2} e^{-ik_2 x} \right) dx$$

$$= \sum_{k_1,k_1} \hat{u}_{k_1} \hat{u}^*_{k_2} \underbrace{\int_0^{2\pi} e^{i(k_1-k_2)x} \, dx}_{2\pi \delta_{k_1 k_2}} = 2\pi \sum_k |\hat{u}_k|^2$$

Then $\|u - P_N u\| = \left(2\pi \sum_{|k|>K} |\hat{u}_k|^2\right)^{1/2}$

$$= \sqrt{2\pi}\left(\sum_{|k|>K} \frac{|k|^{2m}}{|k^{2m}|}\, |\hat{u}_k|^2\right)^{1/2} \le \qquad h = \frac{2\pi}{N} = \frac{\pi}{K}$$

$$\sqrt{2\pi}\left(\frac{h}{\pi}\right)^m \left(\sum_{|k|>K} |k|^{2m}\, |\hat{u}_k|^2\right) \le \sqrt{2\pi}\left(\frac{h}{\pi}\right)^m \times$$

$$\left(\sum_{k} k^{2m}\, |\hat{u}_k|^2\right)^{1/2} = \sqrt{2\pi}\left(\frac{h}{\pi}\right)^m \|\partial_x^m u\|$$

Rate of convergence controlled by the highest bounded derivative! if all der. ex. convergence faster than any power

Fourier transf. is a very fast : FFT

$N \ln N$. Write for the moment

$$\begin{cases} v_j = \sum_{l=0}^{N-1} \hat{v}_l e^{2\pi i l j/N} \\ \hat{v}_l = \frac{1}{N} \sum_{j=0}^{N-1} v_j e^{-2\pi i l j/N} \end{cases}$$

; seems like N^2 op.

Let $N = 2^M$, otherwise pad with zeroes

$$V_\ell = \sum_{j=0}^{N-1} e^{2\pi i j \ell/N} v_j \underset{\text{even + odd}}{\overset{\uparrow}{=}} \sum_{j=0}^{N/2-1} e^{2\pi i (2j)\ell/N} v_{2j}$$

$$+ \sum_{j=0}^{N/2-1} e^{2\pi i (2j+1)\ell/N} v_{2j+1} = \sum_{j=0}^{N/2-1} e^{2\pi i j \ell/(N/2)} v_{2j}$$

do FT on even points

$$+ \underbrace{e^{2\pi i \ell/N}}_{\omega^\ell} \sum_{j=0}^{N/2-1} e^{2\pi i j \ell/(N/2)} = V_\ell^{e} + \omega^\ell V_\ell^{o} \leftarrow \text{odd}$$

two transforms with half as many points! Can iterate until no points left

Genesis of pseudospectral method

Orszag (1970) Idea: $\underbrace{\frac{d^p}{dx^p}u}_{O(N)}$: do in spectral space

$u(x)\,v(x)$: do in real space $O(N)$

Burger's eqn: $u_t + u u_x = \nu u_{xx}$ on $x \in [0, 2\pi]$

Temp. discr:

$$\frac{u^{n+1}(x) - u^n(x)}{\Delta t} + u^n(x)\frac{du^n}{dx} = \nu \underbrace{\frac{d^2}{dx^2}u^{n+1}}_{\text{implicit}}$$

Multiply by e^{-ikx} and integrate

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\,(\overset{\text{Burgers}}{eq})\,dx = 0$$

Then $(\Delta t^{-1} + \nu k^2)\,\hat{u}_k^{n+1} = \Delta t^{-1}\,\hat{u}_k^n - \underbrace{\hat{\omega}_k^n}_{\text{nonlin.}}$

Now $\hat{\omega}_k = \frac{1}{2\pi}\int_0^{2\pi} u\,\frac{du}{dx}\,e^{-ikx}\,dx =$

$$\sum_{p,k=-k} \hat{u}_p\, iq\,\hat{u}_q\, e^{i(p+q)x}$$

$$\sum_{p,q=-K}^{K} \hat{u}_p \, iq \, \hat{u}_q \, \delta_{p+q-k} = \underbrace{\sum_{\substack{p,q=-K \\ p+q=k}}^{K} \hat{u}_p \, iq \, \hat{u}_q}_{\hat{\omega}_k}$$

Do the sum: N^2 steps!

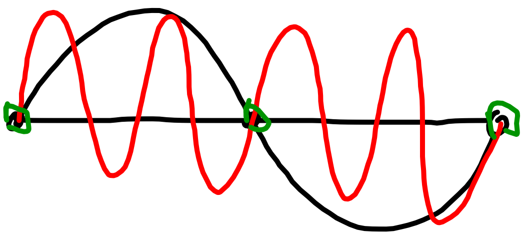
Instead, I do multiplication in real space

$$\tilde{\omega}_k^n = F_k \left\{ u_l^n \left(\frac{du}{dx}\right)_l^n \right\} = F_k \left\{ F^{-1}\{\hat{u}_k^n\} \cdot F_l^{-1}\{ik\hat{u}_k^n\} \right\}$$

Note that $\hat{\omega}_k \neq \underbrace{\tilde{\omega}_k}$ done by Fourier transform

only true up to higher order k-modes,
do to aliasing see Q6, sheet

Idea



On the green grid, the two functions are identical!