§5.3 Time stepping and Stability regions

Time-dependent equation: evaluate derivatives spectrally, use finite differences in time: method of lines.

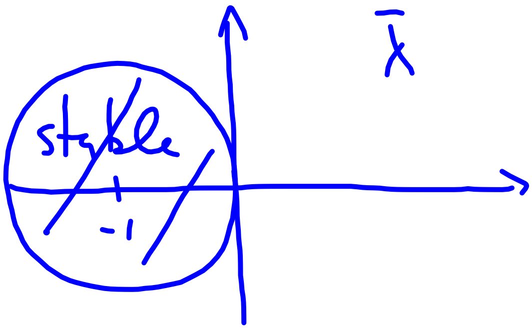
Eigenvalues are $(ik)^p$: sufficient to look at scalar equations

① Euler : $\dfrac{u^{n+1}-u^n}{\Delta t} = \lambda u^n$ $\Big\}$ want $|\lambda|$ large, $\lambda < 0$, so that eqn is stable

solution: $u^n = A z^n$, put $\bar{\lambda} = \lambda \Delta t$

$\rightsquigarrow \boxed{z = 1 + \bar{\lambda}}$

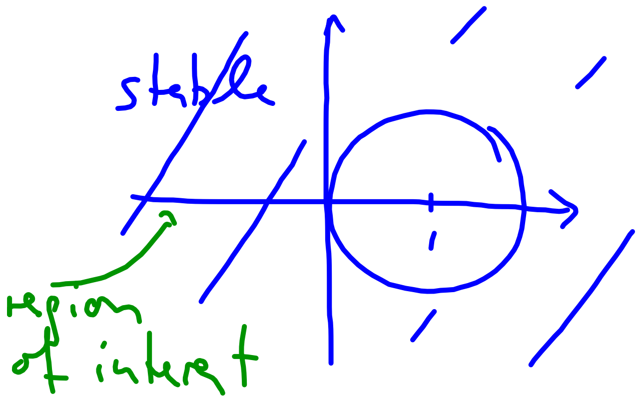
stability: $|z| \leq 1$



if $\bar{\lambda}$ real
stable for $|\bar{\lambda}| < 2$

② Backwards Euler

$$\frac{u^{n+1} - u^n}{\Delta t} = \lambda u^{n+1} \rightsquigarrow z = \frac{1}{1-\bar{\lambda}}$$



unconditionally stable!

③ <u>Leapfrog</u> $\frac{u^{n+1}-u^{n-1}}{2\Delta t} = \lambda u^{n}$

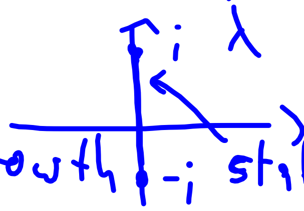
$\rightsquigarrow \boxed{z^2 - 2\bar{\lambda} z - 1 = 0}$ — if z sol., then $-1/z$ also solution!

$\Longleftrightarrow \; 1 + \frac{2\bar{\lambda}}{z} - \frac{1}{z^2} = 0$

for stability $|z_1| = |z_2| = 1$ (if $|z_1| < 1$ $\rightsquigarrow$ $|z_2| > 1$ unstable)

Hence $z = e^{i\theta}$ $\quad z - \frac{1}{z} = 2i\sin\theta = 2\bar{\lambda}$

$\boxed{\bar{\lambda} = i\sin\theta}$ stability region



stable

exclude $\lambda \neq \pm i$, secular growth

Consider $u_t + u_x = 0 \rightsquigarrow \underline{u}_t + \underline{\underline{D}}^{(1)}\underline{u} = 0$

eigenvalues of $D^{(1)}$ are ik, $-K \le k \le K$

Leapfrog: eigenfct

$$\underline{u} = \begin{bmatrix} u(x_1) \\ \vdots \\ u(x_N) \end{bmatrix} = \begin{bmatrix} e^{ikx_1} \\ \\ e^{ikx_N} \end{bmatrix}$$

$[-iK, iK] \subset (-i, i)$

or $\Delta t < \frac{1}{K} = \frac{2}{N-1}$ for stability

worst case: $\lambda = \pm iK$,

eigenmode $e^{\pm iK}$

$$e^{\pm iK \overbrace{h}^{\text{grid spacing}}} = e^{i\left(\frac{N-1}{N}\right)\pi} \approx -1$$

sawtooth mode

§ Chebyshev Spectral Methods

- Apply in bounded domains
  (channel or a pipe)
- still have FFT

Basic properties

(a) defined over $[-1, 1]$,

$$\begin{cases} T_n = \cos n\theta \\ x = \cos\theta \\ |T_n(x)| \le 1 \end{cases}$$

(b) boundary values:

$T_n(1) = 1$

$T_n(-1) = (-1)^n$

$T_0(x) = 1$, $T_1(x) = x$

$T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$

(c) orthogonality $\displaystyle\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx$

show using $x = \cos\theta$

$$= \begin{cases} 0 & m \neq n \\ \pi & m = n = 0 \\ \pi/2 & m = n \neq 0 \end{cases}$$

(d) $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$, $n \geq 1$

(e) $(1-x^2)\dfrac{d^2T_n}{dx^2} - x\dfrac{dT_n}{dx} + n^2 T_n(x) = 0$

$n = 0, 1, 2, \ldots$

(f) Derivatives (use chain rule)

$$\frac{dT_n}{dx} = \frac{n \sin(n \cos^{-1} x)}{\sqrt{1-x^2}}$$

Derivative on the boundary:

L'Hôpital's rule $\frac{dT_n}{dx}(1) = n^2$

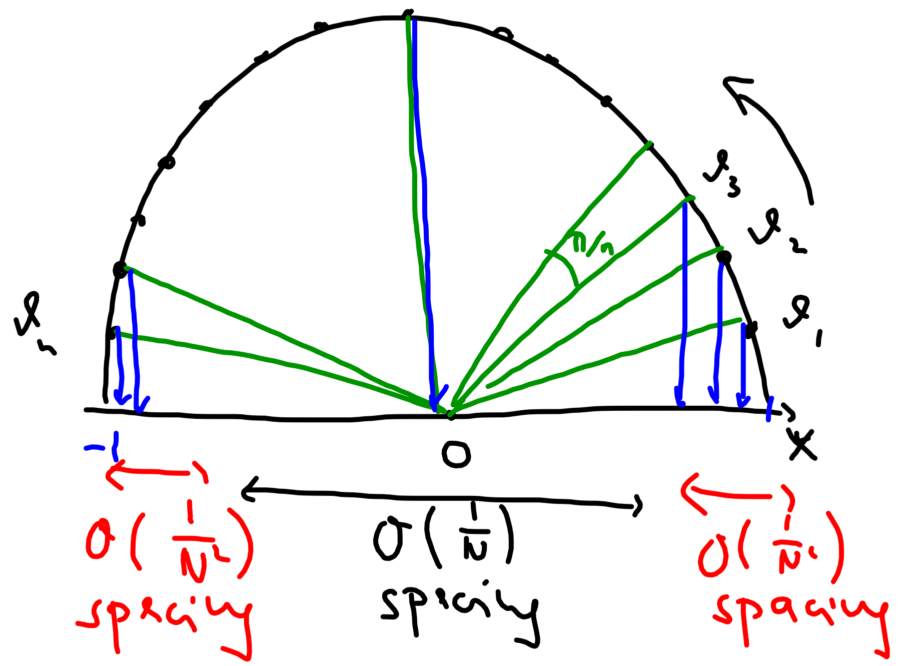
(3) Zeros of Chebyshev polynomials

$T_n(x) = 0 \Rightarrow n \cos^{-1} x_j = \frac{\pi}{2} + \pi(j-1),\ j = 1, 2, \dots n$

$\Rightarrow x_j = \cos\Big(\underbrace{\frac{\pi(2j-1)}{2n}}_{\vartheta_j}\Big)$

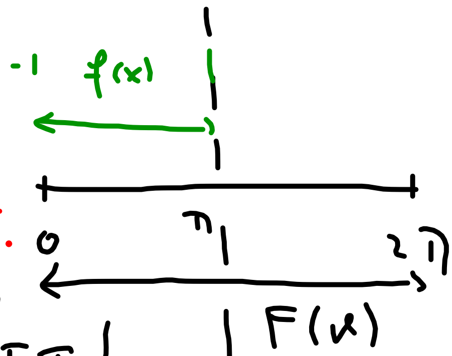
ϑ_j equally spaced around a semicircle

$\rightsquigarrow$ use FFT!



ϑ_3
ϑ_2
ϑ_1
ϑ_n
π/n
-1
0
x
$O(\frac{1}{N^2})$ spacing
$O(\frac{1}{N})$ spacing
$O(\frac{1}{N^2})$ spacing

Interpolating $f(x)$ over $[-1,1]$ at Chebychev zeroes $\equiv$ interpolating $F(\theta) = f(\cos\theta)$ using equally spaced points over $[0,\pi]$



Since $F(-\theta) = F(\theta)$ and $F(\theta \pm 2\pi n) = F(\theta)$ F is 2π-per.

$\rightsquigarrow$ connection to Fourier methods $\rightsquigarrow$ can use FFT!

$F(\theta) = F(2\pi - \theta)$

§ 6.1 1D Boundary Value Problems

Consider $u_{xx} = e^{4x}$ $\quad x \in [-1, 1]$
$\quad u(\pm 1) = 0$

As before, 3 different approaches

(A) work in physical space
⇝ Define differentiation matrix $\underline{\underline{D}}^{(p)}$ in real space, modified to include boundary conditions. see [4] p. 61 for discussion.

Ⓑ Work in spectral space

1) Galerkin projection

note: bound. conditions are not automatically satisfied

Tau method:

Let $u = \sum_{n=0}^{N-1} a_n T_n(x)$

insist that

$$0 = \langle T_m(x), u_{xx} - e^{4x} \rangle = \langle T_m(x), \sum_{n=0}^{N-1} a_n T_n''(x) - e^{4x} \rangle$$

where $\langle A, B \rangle = \int_{-1}^{1} \frac{A\,B}{\sqrt{1-x^2}}\,dx$ for $m = 0, 1, \dots N-3$

⇝ put b.c. in as constraint

⇝ τ method.

Implement b.c. :
$$\left\{ \begin{array}{ll} \sum_{n=0}^{N-1} a_n T_n(1) = 0 & \text{N-1st eqn.} \\ \sum_{n=0}^{N-1} a_n T_n(-1) = 0 & \text{N'th eqn} \end{array} \right.$$

total of N eqn.

$$\begin{bmatrix} \langle T_0, T_0'' \rangle & \langle T_0, T_1'' \rangle & \cdots & \langle T_0, T_{N-1}'' \rangle \\ \langle T_1, T_0'' \rangle & \vdots & \cdots & \\ \vdots & & & \\ T_0(1) & T_1(1) & \cdots & T_{N-1}(1) \\ T_0(-1) & \cdots & & T_{N-1}(-1) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \\ a_{N-1} \end{bmatrix} = \begin{bmatrix} \langle T_0, e^{4x} \rangle \\ \langle T_1, e^{4x} \rangle \\ \vdots \\ \langle T_{N-3}, e^{4x} \rangle \\ 0 \\ 0 \end{bmatrix}$$

upshot: $\underline{\underline{A}}\,\underline{a} = \underline{b}$

- con: matrix is dense
- expect exponential convergence for smooth function

2) <u>Collocation</u> again represent

$$u = \sum_{n=0}^{N-1} a_n T_n(x)$$

Now we require that

$$\sum a_n T_n(1) = 0 \quad j = 1$$

$$\sum a_n T_n''(x_j) - e^{4x_j} = 0 \quad j = 2, \dots, N-1$$

$$x_j = \cos\left[\frac{\pi(2j-1)}{2N}\right]$$

$$\sum a_n T_n(-1) = 0 \quad j = N$$

$\rightsquigarrow$ the matrix problem becomes

$$
\text{b.c.} \quad
\begin{bmatrix}
T_0(1) & T_1(1) & \cdots & T_{N-1}(1) \\
\hline
T_0''(x_2) & & \cdots & T_{N-1}''(x_2) \\
\vdots & & & \vdots \\
T_0''(x_{N-1}) & & \cdots & T_{N-1}''(x_{N-1}) \\
\hline
T_0(-1) & T_1(-1) & \cdots & T_{N-1}(-1)
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_{N-1}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ e^{4x_2} \\ e^{4x_3} \\ \vdots \\ e^{4x_{N-1}} \\ 0
\end{bmatrix}
\quad \text{b.c.}
$$

- very simple to program (+)
- full matrix (-)
- exponential convergence

<u>Another approach</u>: redefine expansion

<u>For example</u>: $\phi_n(x) = T_n(x) - T_{n-2}(x) \quad n \geq 2$

$\leadsto$ so $\phi_n(\pm 1) = 0$ b.c. are satisfied identically!

or (see [3] p 113)

$\phi_{2n} := T_{2n}(x) - 1$ (zero on ± 1)

$\phi_{2n+1} := T_{2n+1}(x) - x$ (")

plus: don't have to worry about b.c.

minus: not clear convergence to exponential