NUMERICAL METHODS FOR PDES: PROBLEM SHEET 1

Abstract. This sheet covers numerical differentiation, interpolation and illustrates the concepts of consistency, stability and convergence in the simpler context of numerical methods for ordinary differential equations.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function of the variable x. Write down an approximation to the second derivative $f''(x_j)$ in terms of the values of f at the following points.
 (a) $x_j - h$, x_j and $x_j + h$ (centered formula),
 (b) $x_j - 2h$, $x_j - h$ and x_j (left-sided formula).
 (c) x_j, $x_j + h$ and $x_j + 2h$ (right-sided formula).
 (d) x_j, $x_j + h, x_j + 2h$ and $x_j + 3h$ (right-sided formula).
 (e) $x_j - \lambda h$, x_j and $x_j + h$ (non-uniform grid)

 In each case, use two approaches: (i) the Taylor series approach and (ii) the interpolating polynomial approach. Also, state the order of accuracy of the formula in the limit $h \to 0$.

(2) This exercise is taken from [1], p. 44. Examine how polynomial interpolation over a uniform grid can go wrong by experimenting with the following MATLAB code which considers $f(x) := 1/(1 + 16x^2)$ over $[-1, 1]$. Try $N = 5, 10, 15, 20$, as well as other functions which are smooth in the real line segment $[-1, 1]$ but have singularities nearby in the complex plane. Then try another function which is analytic like $f(x) := e^x$ (although MATLAB's interpolation function may struggle if N is too large!). Verify that interpolation with respect to Chebyshev points always works well.

```
% Matlab program: Polynomial Interpolation
N=10;
xx=-1.01:0.005:1.01;
for i=1:2
if i==1, s='equispaced pts'; x=-1+2*(0:N)/N; end
if i==2, s='Chebyshev pts';  x=cos(pi*(0:N)/N); end
subplot(2,1,i)
% change function in the next two lines
u =1./(1+16*x.^2);
uu=1./(1+16*xx.^2);
p=polyfit(x,u,N);     % calculate interpolating poly.
pp=polyval(p,xx);     % evaluate poly over dense grid
% plot interpolant over equispaced grid
plot(x,u,'.b','markersize',13)
hold on
plot(xx,pp,'-b')
plot(xx,uu,'-r')
axis([-1.1 1.1 -1 1.5]); title(s)
error=norm(uu-pp,inf);
```

```
text(-0.5,-0.5,['max error =' num2str(error)])
end
```

(3) If L is a nonzero integer then the initial value ODE problem

$$u_t(t) = f(u,t) := \frac{L}{t+1}u(t), \qquad u(0) = 1$$

has a unique solution $u(t) = (t+1)^L$. Suppose we calculate an approximation to $u(2)$ using the following methods:
(a) $u^{n+1} = u^n + kf^n$ (Euler's Method)
(b) $u^{n+1} = u^{n-1} + 2kf^n$ (Midpoint rule)
(c) $u^{n+1} = u^n + \frac{k}{24}(55f^n - 59f^{n-1} + 37f^{n-2} - 9f^{n-3})$ (higher order Adams-Bashforth method)
(using exact values where needed for initialisation). Find the order of accuracy of the methods and hence identify the maximum value of L for which each method will be *exact*.

(4) By the method of undetermined coefficients, derive a third-order finite-difference method of the form

$$u^{n+1} = \alpha u^n + \beta u^{n-1} + k(\gamma f^n + \delta f^{n-1})$$

for integrating the ODE $u_t = f(u,t)$.
Help: You should find

$$u^{n+1} = -4u^n + 5u^{n-1} + k(4f^n + 2f^{n-1})\,.$$

Then show using "von Neumann analysis", i.e. by looking for solutions of the form

$$u^n = g^n \quad (g \text{ to the power } n)$$

that the formula is unstable in the limit as $k \to 0$ and therefore useless.

(5) Consider the finite-difference method

$$u^{n+1} = 2u^n - u^{n-1}$$

for the ODE $u_t = f(u,t)$. Why is this method flawed? Show that it is consistent but unstable.

(6) Which of the following formulae for integrating the ODE $u_t = f(u,t)$ are convergent? Are the nonconvergent ones inconsistent or unstable or both?
(a) $u^{n+1} = \frac{1}{2}u^n + \frac{1}{2}u^{n-1} + 2kf^n$
(b) $u^{n+1} = u^n$
(c) $u^{n+4} = u^n + \frac{4}{3}k(f^{n+3} + f^{n+2} + f^{n+1})$
(d) $u^{n+3} = u^{n+1} + \frac{1}{3}k(7f^{n+2} - 2f^{n+1} + f^n)$

References

1. Lloyd. N. Trefethen, *Spectral Methds in Matlab*, SIAM, Philadelphia, 2000.