NUMERICAL METHODS FOR PDES: PROBLEM SHEET 2

ABSTRACT. This sheet focuses on the case where the partial differential equation is of parabolic type.

(1) Consider the heat equation

$$u_t = \frac{\sigma^2}{2} u_{xx}\,, \quad t > 0\,,$$

subject to the initial condition $u(\cdot, 0) = u_0$.

(a) Show that it is well-posed as a pure initial-value problem on the real line, in the space of square-integrable functions.

(b) Show that it is well-posed as an initial-boundary-value problem on the interval $0 < x < L$, with periodic boundary conditions, in the space of square-integrable L-periodic functions.

(2) In this question, we review some properties of the discrete Fourier transform that were used in the lectures to justify von Neumann analysis.

Let $X := \mathbb{C}^N$ be equipped with the norm

$$\|v\| := \left(h \sum_{j=0}^{N-1} |v_j|^2 \right)^{\frac{1}{2}}$$

where $h = L/N$ and L is some fixed positive number. We put

$$x_j = jh\,, \quad 0 \le j < N\,,$$

and define the *discrete Fourier transform* $\widehat{v}$ of v as the vector in $\mathbb{C}^N$ with components

$$\widehat{v}_\ell = \frac{1}{N} \sum_{j=0}^{N-1} v_j \, \mathrm{e}^{-\mathrm{i}\xi_\ell x_j} \quad \text{where} \quad \xi_\ell h = \frac{\ell}{N} 2\pi\,, \ 0 \le \ell < N\,.$$

(a) Show that, for $0 \le j < N$,

$$v_j = \sum_{\ell=0}^{N-1} \widehat{v}_\ell \, \mathrm{e}^{\mathrm{i}\xi_\ell x_j}\,.$$

(b) Prove *Parseval's identity*: For every u, $v \in X$,

$$h \sum_{j=0}^{N-1} u_j \, \overline{v_j} = L \sum_{\ell=0}^{N-1} \widehat{u}_\ell \, \overline{\widehat{v}_\ell}\,.$$

(c) Deduce that

$$\|v\|^2 = L \sum_{\ell=0}^{N-1} |\widehat{v}_\ell|^2\,.$$

(3) Consider the following finite difference methods to integrate the heat equation $u_t = u_{xx}$. Here, $\mu := k/h^2$ where k denotes the time step and h the space step. In each case, find the order of accuracy and use von Neumann analysis to determine the range of μ for which the method is stable.

(a) $u_j^{n+1} = u_j^n + \mu(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$ (Euler's Method)

(b) $u_j^{n+1} = u_j^n + \mu(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})$ (Backward Euler)

(c) $u_j^{n+1} = u_j^n + \frac{1}{2}\mu(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \frac{1}{2}\mu(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})$ (Crank-Nicolson method)

(d) $u_j^{n+1} = u_j^{n-1} + 2\mu(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$ (Leapfrog method)

[**Note** the general form

$$u_j^{n+1} = u_j^n + (1-\theta)\mu(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + \theta\mu(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1})$$

for $0 \leq \theta \leq 1$ has the special cases $\theta = 0$ (Euler's method), $\theta = 1/2$ (Crank-Nicolson) and $\theta = 1$ (backward Euler)].

(4) For solving the nonlinear heat equation $u_t = (a(u)u_x)_x$ consider the explicit scheme

$$u_j^{n+1} = u_j^n + \mu[\, a_{j+1/2}^n(u_{j+1}^n - u_j^n) - a_{j-1/2}^n(u_j^n - u_{j-1}^n)\,]$$

where $a_{j\pm 1/2}^n := \frac{1}{2}[\, a(u_{j\pm 1}^n) + a(u_j^n)\,]$ and $\mu := k/h^2$. If $0 < a_* \leq a(u) \leq a^*$, apply von Neumann analysis by freezing the nonlinear coefficient in order to determine a plausible condition, in terms of μ, for stability.

(5) Consider the advection-diffusion equation

$$u_t + au_x - u_{xx} = 0$$

with Dirichlet boundary conditions and $a \geq 0$ a constant. Show that

$$u(x,t) = \exp[-(\mathrm{i}\ell\pi a + \ell^2\pi^2)t + \mathrm{i}\ell\pi x]$$

is, for every ℓ, a particular solution of the problem. Use von Neumann analysis to derive the stability condition for the upwind scheme

$$\frac{u_j^{n+1} - u_j^n}{k} + a\frac{u_j^{n+1} - u_{j-1}^{n+1}}{h} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2}.$$

What is the order of accuracy of the scheme?