NUMERICAL METHODS FOR PDES: PROBLEM SHEET 3

Abstract. This sheet is devoted to the hyperbolic case.

(1) The motion of shallow layer of fluid is modelled by the equations

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0\,.$$

Here, h is the height of the layer, u the fluid's velocity, and g the gravitational acceleration near the earth's surface. Under what condition is this system hyperbolic? Write down the equations for its characteristic curves.

(2) Let u be the solution of the partial differential equation

$$u_t + au_x = 0$$

where a is a positive constant. Determine the coefficients c_0, c_1 and c_{-1} so that the order of consistency of the scheme

$$u_j^{n+1} = c_{-1}u_{j-1}^n + c_0 u_j^n + c_1 u_{j+1}^n$$

is highest. Verify that the result is the Lax-Wendroff scheme. Show using von Neumann analysis that the CFL condition is also *sufficient* for this scheme.

(3) For the linear advection equation $u_t + au_x = 0$ (a a positive constant), a generalised upwind scheme on a uniform mesh is defined by

$$u_j^{n+1} = (1-\theta)u_k^n + \theta u_{k-1}^n$$

where $x_k - \theta h = x_j - ak$ and $0 \le \theta < 1$. Verify that the CFL condition places no restriction on k and that von Neumann stability analysis also shows that stability is unrestricted. What is the order of accuracy of the scheme?

(4) Consider the wave equation for the solution of

$$u_{xx} - (1+4x)^2 u_{tt} = 0$$

in the region $0 < x < 1$ and $t > 0$, subject to

$$u(x,0) = x^2, \quad u_t(x,0) = 0, \quad u_x(0,t) = 0, \quad u(1,t) = 1.$$

(a) Rewrite the equation as a system of two first-order equations. Hence find the characteristics.
(b) Derive an explicit central difference scheme for the equation, i.e. one that uses the three spatial grid points x_{j-1}, x_j and x_{j+1}.
(c) Use the CFL condition to derive a necessary condition for stability.

(5) The leap frog scheme for $u_t = u_x$ is

$$u_j^{n+1} = u_j^{n-1} + \mu(u_{j+1}^n - u_{j-1}^n)$$

where $\mu := k/h$. For $\mu = 0.9$, $u = 0$ at the right-hand boundary and initial data exact, consider the two possible left-hand boundary conditions: a) $u_0^{n+1} = u_1^n$ and b) $u_0^{n+1} = u_1^{n+1}$. Discuss which is likely to be stable and which unstable.

(6) Consider the implicit scheme

$$u_j^{n+1} = u_j^n - \frac{\mu}{2}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right), \quad \mu := ak/h,$$

for solving $u_t + au_x = 0$ where a is a constant.

(a) Show that the amplification factor $g = g(\xi)$ in the von Neumann analysis of the scheme is

$$\frac{1}{1 + \mathrm{i}\mu \sin \xi h}.$$

Deduce that the scheme is unconditionally stable.

(b) Describe the corresponding dissipation and dispersion errors for the scheme.

(7) Let $\mu := k/h$. Determine the numerical dispersion relation for the following discretisations of $u_t = u_x$. In each case, determine the leading behaviour of the dispersion error for small wave numbers.

(a) The Lax–Wendroff scheme

$$u_j^{n+1} = u_j^n + \frac{1}{2}\mu(u_{j+1}^n - u_{j-1}^n) + \frac{1}{2}\mu^2(u_{j+1}^n - 2u_j^n + u_{j-1}^n).$$

(b) The upwind scheme

$$u_j^{n+1} = u_j^n + \mu(u_{j+1}^n - u_j^n).$$

(c) The leapfrog scheme

$$u_j^{n+1} = u_j^{n-1} + \mu(u_{j+1}^n - u_{j-1}^n).$$