NUMERICAL METHODS FOR PDES: PROBLEM SHEET 4

Abstract. Problems for the second half of the course

(1) **Elliptic PDEs**. Consider the 1D BVP $u_{xx} = f(x)$ with $u(0) = 0$ and $u(1) = 1$. Write down the finite set of equations which u_j must solve if a uniform grid with $h = 1/N$ is used to discretise the equation and centred differences are used. Show that the discretized system admits the exact solution when $f(x) := 2$ and therefore solves the problem exactly *for all* N. Explain why and discuss for what other choices of $f(x)$ this also holds true. In 2D, so $u_{xx} + u_{yy} = f(x, y)$, which $f(x, y)$ will also give this result?

(2) **Conjugate Gradient Method**. Solve the linear system

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

using the Conjugate Gradient Method. Using the characteristic polynomial of A

$$A^3 - 6A^2 + 10A - 4I = 0,$$

write down the answer as a sum of matrix multiplications involving A and check that this coincides with your conjugate gradient result.

(3) **Fourier Spectral Methods**. Consider the heat equation $u_t = u_{xx}$ over $[0, 2\pi]$ and initial condition

$$u(x, 0) = \begin{cases} 0 & x \neq \pi, \\ 1 & x = \pi. \end{cases}$$

Formally we could adopt the boundary conditions $u_x(0, t) = u_x(2\pi, t) = 0$ or $u(0, t) = u(2\pi, t) = 0$ to complete the problem specification but given the (severely) localised initial conditions and short enough times, these are not that important. Also we don't expect the solution to be 2π-periodic for subsequent times but it will so close to zero at the ends that it can be regarded as periodic in practice. So solve this problem using a Fourier Spectral method based on the uniform grid $x_j := 2\pi j/N$ for $j = 0, 1, 2, ..., N$ to find

$$u_j(t) = \frac{1}{N} \sum_{k=-K}^{K} e^{ik(x_j - \pi) - k^2 t} \qquad (*)$$

where $N = 2K+1$ (HINT: to make this manageable adopt the simplification that one of the grid points is precisely at π). Confirm this satisfies the heat equation (to get the i.c. to work will require reversing the simplification!). Note also that the sum (*) at $t = 0$ does not show any spectral drop off. Try a smoother initial condition and convince yourself that the initial sum will then show this drop off.

(4) **Fourier Spectral Methods**. Consider Burgers' equation

$$u_t + uu_x = \nu u_{xx}.$$

Develop a Fourier spectral method in physical space, with the Crank-Nicolson method used for time stepping. Investigate the stability using von Neumann analysis, using a frozen coefficient assumption: $u_l^n \equiv \bar{u}$ in the nonlinear term, and determine the order of the scheme.

(5) **Fourier Spectral Methods**. Let $u(x) = \cos(mx)$ for some integer m. Calculate uu_x by taking a discrete Fourier Transform

$$(\widehat{uu_x})_k := \sum_{p,q=-K:p+q=k}^{K} \widehat{u}_p (\widehat{u_x})_q = \sum_{p,q=-K:p+q=k}^{K} iq\, \widehat{u}_p \widehat{u}_q \,,$$

simplify this convolution and then take the inverse Fourier Transform. Confirm your result by evaluating the product directly.

Spectral codes actually work in the *reverse* direction! One works in spectral space with $\widehat{u}_k$ and one needs $(\widehat{uu_x})_k$. The approach is then to inverse transform $\widehat{u}_k$ and $ik\widehat{u}_k$ back to physical space to get u and u_x (respectively), evaluate uu_x by (simple) multiplication and then Fourier transform this back to spectral space avoiding a costly convolution computation. Run through this.

(6) **Fourier Spectral Methods**. As shown in the preceding problem, $\hat{w}_k := (\widehat{uu_x})_k$ can be calculated as a sum, with a numerical effort of $O(N^2)$. A much more efficient method is to calculate the same quantity using the FFT. Let

$$\tilde{w}_k = F\left\{F_l^{-1}\left\{\hat{u}_k\right\} F_l^{-1}\left\{ik\hat{u}_k\right\}\right\}$$

be the approximation to $\hat{w}_k$ thus obtained. Calulcate the aliasing error $\epsilon_k = \tilde{w}_k - \hat{w}_k$, and give an explicit expression for ϵ_k for the case $u(x) = \cos(mx)$, for some integer m.

(7) **Chebyshev Polynomials**. Show that

(a) $T_{2n}(x) = T_n(2x^2 - 1) = 2T_n(x)^2 - 1$

This identity can be used to calculate even Chebyshev polynomials only, with the argument x replaced by $2x^2 - 1$.

(b) $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$

This recursion is used to calculate $T_n(x)$ at a given point x recursively.

(c) $\dfrac{dT_n(x)}{dx} = \dfrac{n \sin n\theta}{\sin\theta}$ where $x = \cos\theta$

(d) $\dfrac{dT_n^2(x)}{dx^2} = \dfrac{n \sin n\theta \cos\theta}{\sin^3\theta} - \dfrac{n^2 \cos n\theta}{\sin^2\theta}$ where $x = \cos\theta$

(8) **Chebyshev Spectral Methods**. Consider the eigenvalue problem

$$u_{xx} + \lambda u = 0$$

over $[-1, 1]$ with $u(\pm 1) = 0$ where λ is the eigenvalue. Solve this problem using Chebyshev spectral methods adopting the following different strategies.

(a) Use a Tau method.

(b) Use a collocation method explicitly imposing the b.c.s.
(c) Use a collocation method implicitly imposing the b.c.s (i.e. build the b.c.s into the spectral functions: e.g. use $\phi_n := T_{n+2}(x) - T_n(x)$; or $\phi_n := (1 - x^2)T_n(x)$; or $\phi_n := T_n(x) - 1$ (n even) and $\phi_n := T_n(x) - T_1(x)$ (n odd)).