Sheet 1 : solutions

(1) In all those problems, the idea is to seek a linear combination of the function values that makes the difference with $f''(x_j)$ as small as possible in the limit as $h \to 0$. (Taylor expansion approach) or the compute the second derivative of the appropriate interpolant. The resulting formulae, are

(a) $$\frac{1}{h^2}\left[ f(x_{j-1}) - 2f(x_j) + f(x_{j+1}) \right] = f''(x_j) + O(h^2)$$

(b) $$\frac{1}{h^2}\left[ f(x_j - 2h) - 2f(x_j - h) + f(x_j) \right] = f''(x_j) + O(h)$$

(c) $$\frac{1}{h^2}\left[ f(x_j) - 2f(x_j + h) + f(x_j + 2h) \right] = f''(x_j) + O(h)$$

(d) $$\frac{1}{h^2}\left[ 2f(x_j) - 5f(x_j + h) + 4f(x_j + 2h) - f(x_j + 3h) \right] = f''(x_j) + O(h^2)$$

(e) $$\frac{1}{h^2}\left[ \frac{2}{\lambda(\lambda+1)} f(x_j - \lambda h) - \frac{2f(x_j)}{\lambda} + \frac{2}{1+\lambda} f(x_j + \lambda h) \right] = f''(x_j) + \begin{cases} O(h^2) & \text{if } \lambda = 1 \\ O(h) & \text{otherwise} \end{cases}$$

(3) The idea is to determine the order of accuracy $p$ of each method. For $L$ a natural number, the method is exact whenever $p \geq L$. This is because, for this particular differential problem $u^{(r)} = 0$ for every $r > L$. We illustrate this with Euler's method in (a). Here $p = 1$ and so the method is exact when $L = 1$. Indeed, the sequence

$$u^n = t_n + 1$$

satisfies

$$u^{n+1} = t_{n+1} + 1 = t_n + 1 + k = u_n + \frac{k\, u_n}{1 + t_n}$$

On the other hand

$$u^n = (t_n + 1)^2$$

satisfies

$$u_{n+1} = (t_{n+1}+1)^2 = (t_n+1) + 2k(t_n+1) + k^2$$

$$= u_n + \frac{k\,2}{1+t_n} u_n + k^2$$

and so is <u>not</u> a solution of the discrete equation for $L=2$.

(4) von Neumann analysis cannot be applied, strictly speaking, unless $f$ is a linear function of $u$. So what we do is to replace $f$ by, say, $\lambda u$ so that the difference equation becomes

$$u^{n+1} = -4u^n + 5u^{n-1} + \lambda k\left(4u^n + 2u^{n-1}\right)$$

The substitution $u^n = g^n$ ($g$ to the power of $n$) then leads to

$$g^2 + 4(1+\lambda k)g - 5 - 2\lambda k = 0$$

For $k=0$, this becomes $(g+5)(g-1) = 0$

We conclude that, for $k$ small, one root is close to $-5$. So the scheme is unstable.

(5) The discretisation uses no information about $f$. We have

$$L^n = u(t_{n+1}) - 2u(t_n) + u(t_{n-1})$$

$$= [\cancel{u(t_n)} + \cancel{ku'(t_n)} + O(k^2)] - \cancel{2u(t_n)} + [\cancel{u(t_n)} - \cancel{ku'(t_n)} + O(k^2)]$$

$$= O(k^2) \quad \text{as } k \to 0$$

so the order of accuracy is 1. To study stability, put $u^n = g^n$. Then

$$g^2 - 2g + 1 = 0$$

So $g=1$ is a root of multiplicity 2. The general solution of the difference equation is

$$u_n = An + B$$

Hence instability.

(1) (a) Consistency: $u(t_{n+1}) - \frac{1}{2}u(t_n) - \frac{1}{2}u(t_{n-1}) - 2k\,u_t(t_n)$

$= \left[\cancel{u(t_n)} + k u_t(t_n) + O(k^2)\right] - \cancel{\frac{1}{2}u(t_n)} - \frac{1}{2}\left[\cancel{u(t_n)} - k u_t(t_n) + O(k^2)\right] - 2k u_t(t_n)$

$= O(k)$

So it is inconsistent. Setting $f = \lambda u$ and $u^n = g^n$, we find

$$g^2 = \left(\frac{1}{2} + 2k\lambda\right) g + \frac{1}{2}$$

When $k=0$, this gives

$$0 = g^2 - \frac{1}{2}g - \frac{1}{2} = (g-1)(g+\tfrac{1}{2})$$

So the linearisation is stable.

(b) Inconsistent but stable.

(c) Consistency: $u(t_{n+4}) - u(t_n) - \frac{4}{3}k\left[u_t(t_{n+3}) + u_t(t_{n+2}) + u_t(t_{n+1})\right]$

$= \left[\cancel{u(t_n)} + \cancel{4k u_t(t_n)} + O(k^2)\right] - \cancel{u(t_n)} - \frac{4}{3}k\left[\cancel{3u_t(t_n)} + 6k\,u_{tt}(t_n) + O(k^2)\right]$

$= O(k^{p+1})$

with $p \geq 1$. So consistent. To study the stability, we linearise, i.e. put $f = \lambda u$. Then, von Neumann analysis gives

$$g^4 = 1 + \frac{4}{3}k\left(g^3 + g^2 + g\right) \qquad (*)$$

When $k=0$, we have the four roots $g_1 = 1$, $g_2 = -1$, $g_3 = i$, $g_4 = -i$. For $k$ small, these four roots remain close to those values. For instance, the root close to 1 has an expansion of the form

$$g_1(k) = 1 + \alpha_1 k + \beta_1 k^2 + \ldots$$

and substitution into $(*)$ yields

$$1 + 4\alpha_1 k = 1 + \frac{4}{3}k\left(1 + 3\alpha_1 k + 1 + 2\alpha_1 k + 1 + \alpha_1 k\right) + O(k^2)$$

Hence $\alpha_1 = k$.

Likewise, all the other roots deviate by an amount of $O(k)$. The linearised scheme is therefore stable. Convergent

(e) Consistency: $u(t_{n+3}) - u(t_{n+1}) - \frac{k}{3}\left[7u_t(t_{n+2}) - 2u_t(t_{n+1}) + u_t(t_n)\right]$

$= \left[\cancel{u(t_n)} + \cancel{3k u_t(t_n)} + O(k^2)\right] - \left[\cancel{u(t_n)} + \cancel{k u_t(t_n)} + O(k^2)\right]$

$- \frac{k}{3}\left[\cancel{6u_t(t_n)} + 12k\,u_{tt}(t_n) + O(k^2)\right] = O(k^{p+1})$ with $p \geq 1$

So it is consistent.

To investigate stability, we linearise, i.e. we put $f = \lambda u$. Then

$$g^3 - g + \frac{k}{3}\left[7g^2 - 2g + 1\right]$$

For $k=0$, there are three roots: $g_1 = -1$, $g_2 = 0$, $g_3 = 1$ and for small values of $k$ they deviate from these values by an amount $O(k)$.
So the linearised scheme is again stable. Convergent