Sheet 2 : solutions

(1) (a) The solution of the pure initial-value problem is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\, e^{-\sigma^2\xi^2/2\,t + i\xi x}\, d\xi$$

where f is the Fourier transform of the initial datum u_0. This formula says that

$$f(\xi)\, e^{-\sigma^2\xi^2/2\,t}$$

is the Fourier transform of $u(x,t)$. Parseval's identity then says that

$$\int_{-\infty}^{\infty} |u(x,t)|^2\, dx = 2\pi \int_{-\infty}^{\infty} |f(\xi)|^2\, e^{-\sigma^2\xi^2 t}\, d\xi$$

$$\leq 2\pi \int_{-\infty}^{\infty} |f(\xi)|^2\, d\xi = \int_{-\infty}^{\infty} |u_0(x)|^2\, dx$$

Hence

$$\| u(\cdot,t) \| \leq \| u_0 \|$$

where $\| v \|^2 = \int_{-\infty}^{\infty} |v(x)|^2\, dx$

(b) When using periodic boundary conditions, the method of solution is analogous to that for the pure initial-value problem; the Fourier integral is replaced by a Fourier series.

$$u(x,t) = \sum_{n\in\mathbb{Z}} f_n\, e^{-\frac{\sigma^2}{2}\xi_n^2 t}\, e^{i\xi_n x}, \quad \xi_n = \frac{2n\pi}{L}$$

where the f_n are the Fourier coefficients of u_0. The proof is then as in (a).

(2) (a) By definition of $\hat{v}_\ell$, we have

$$\sum_{\ell=0}^{N-1} \hat{v}_\ell\, e^{i\xi_\ell x_j} = \sum_{\ell=0}^{N-1} \left\{ \frac{1}{N}\sum_{m=0}^{N-1} v_m e^{-i\xi_\ell x_m} \right\} e^{i\xi_\ell x_j}$$

$$= \sum_{m=0}^{N-1} v_m \frac{1}{N} \sum_{l=0}^{N-1} e^{il\frac{2\pi}{N}(j-m)}.$$

The inner sum is of the form $\sum_{l=0}^{N-1} z^l$ with $z = e^{i\frac{2\pi}{N}(j-m)}$, and so it is readily seen that

$$\frac{1}{N}\sum_{l=0}^{N-1} e^{il\frac{2\pi}{N}(j-m)} = \begin{cases} 1 & \text{if } j = m \\ 0 & \text{otherwise} \end{cases}$$

Hence the result.

(b) We have

$$h\sum_{j=0}^{N-1} u_j \overline{v_j} = h\sum_{j=0}^{N-1}\left\{ \sum_{l=0}^{N-1} \hat{u}_l e^{i\xi_l x_j} \overline{\sum_{m=0}^{N-1} \hat{v}_m e^{-i\xi_m x_j}} \right\}$$

$$= \sum_{l=0}^{N-1}\sum_{m=0}^{N-1} \hat{u}_l \overline{\hat{v}_m} \underbrace{h\sum_{j=0}^{N-1} e^{ij(l-m)\frac{2\pi}{N}}}_{= Nh \text{ if } l=m \text{ and } 0 \text{ otherwise}}$$

(c) Put $u = v$ in Parseval's identity.

(3) (a) Done in the lectures. The scheme is first-order accurate in time and second-order accurate in space. The stability condition is $\mu \le 1/2$.

(b) Put $L_j^n := u(x_j, t_{n+1}) - u(x_j, t_n) - \mu\left[u(x_{j+1}, t_{n+1}) - 2u(x_j, t_{n+1}) + u(x_{j-1}, t_{n+1})\right]$
and expand at (x_j, t_{n+1}):

$$L_j^n = u(x_j, t_{n+1}) - \left[u(x_j, t_{n+1}) - k u_t(x_j, t_{n+1}) + \frac{k^2}{2} u_{tt}(x_j, t_{n+1}) + \dots \right]$$
$$- \mu\left[h^2 u_{xx}(x_j, t_{n+1}) + O(h^4)\right] = O(k^2 + kh^2)$$

The order of accuracy is the same as in (a).

von Neumann analysis: put $u_j^n = g^n e^{i\xi x_j}$, $\xi h \in [0, 2\pi]$. Then

$$(g^{n+1} - g^n) e^{i\xi jh} = \mu g^{n+1}\left[e^{i\xi h} - 2 + e^{-i\xi h}\right] e^{i\xi jh}$$

Hence, after simplification,

$$g = \frac{1}{1 + 4\mu \sin^2 \xi h/2}$$

So the scheme is unconditionally stable.

(c) The scheme is an average of the Forward and Backward Euler schemes.
We put

$$L_j^h := u(x_j,t_{n+1}) - \tfrac{1}{2}\mu\{u(x_{j+1},t_{n+1}) - 2u(x_j,t_{n+1}) + u(x_{j-1},t_{n+1})$$
$$+ u(x_{j+1},t_n) - 2u(x_j,t_n) + u(x_{j-1},t_n)\} - u(x_j,t_n)$$

Expand at x_j and $t_{n+1/2} := \frac{t_n+t_{n+1}}{2}$ and note that

$$\frac{f(t_n)+f(t_{n+1})}{2} = \frac{1}{2}\Big[f(t_{n+1/2}) + \frac{k}{2}f_t(t_{n+1/2}) + \frac{k^2}{8}f_{tt}(t_{n+1/2}) + \frac{k^3}{48}f_{ttt}(t_{n+1/2}) + \cdots$$

$$+ f(t_{n+1/2}) - \frac{k}{2}f_t(t_{n+1/2}) + \frac{k^2}{8}f_{tt}(t_{n+1/2}) - \frac{k^3}{48}f_{ttt}(t_{n+1/2}) + \cdots\Big]$$

$$= f(t_{n+1/2}) + \frac{k^2}{8}f_{tt}(t_{n+1/2}) + O(k^4)$$

whilst

$$f(t_{n+1}) - f(t_n) = k f_t(t_{n+1/2}) + \frac{k^3}{24}f_{ttt}(t_{n+1/2})$$

Hence $L_j^h = k u_t(x_j,t_{n+1/2}) + \frac{k^3}{24}u_{ttt}(x_j,t_{n+1/2}) + \cdots$

$$-\mu\{u(x_{j+1},t_{n+1/2}) - 2u(x_j,t_{n+1/2}) + u(x_{j-1},t_{n+1/2})\}$$

$$-\frac{\mu k^2}{8}\{u_{tt}(x_{j+1},t_{n+1/2}) - 2u_{tt}(x_j,t_{n+1/2}) + u_{tt}(x_{j-1},t_{n+1/2})\}$$

$$= \cancel{k u_t(x_j,t_{n+1/2})} + \frac{k^3}{24}u_{ttt}(x_j,t_{n+1/2}) - \mu\{\cancel{h^2 u_{xx}(x_j,t_{n+1/2})} + O(h^4)\}$$

$$-\frac{\mu k^2}{8}\{h^2 u_{xxtt}(x_j,t_{n+1/2}) + O(h^4)\} = O(k^3 + kh^2)$$

The scheme is therefore second-order accurate in space and time.
von Neumann analysis yields

$$g = \frac{1 - 2\mu\sin^2 \xi h/2}{1 + 2\mu\sin^2 \xi h/2}$$

Unconditionally stable.

(d) Put $L_j^h = u(x_j,t_{n+1}) - u(x_j,t_{n-1}) - 2\mu\left[u(x_{j+1},t_n) - 2u(x_j,t_n) + u(x_{j-1},t_n)\right]$
Expand at (x_j,t_n):

We find

$$L_j^n = 2k u_t(x_j,t_n) - \frac{k^3}{3} u_{ttt}(x_j,t_n) - \frac{2k}{h^2}\left[h^2 u_{xx}(x_j,t_n) + \frac{h^4}{12} u_{xxxx}(x_j,t_n) + \cdots \right]$$

$$= \frac{k^3}{3} u_{ttt}(x_j,t_n) - \frac{kh^2}{6} u_{xxxx}(x_j,t_n)$$

So the scheme is second-order accurate in both time and space. von Neumann analysis yields

$$g^2 + 2\mu g \sin^2 \frac{\xi h}{2} - 1 = 0$$

The sum of the roots is strictly negative, and their product is 1. So the scheme is unstable.

(4) If a is constant, then the scheme is simply the Forward Euler scheme with $\mu = ak/h^2$. The stability condition is therefore

$$k \leq \frac{1}{2ah^2}$$

(5) We put $u_j^n = g^n e^{i\xi x_j}$. Then, after simplification, we find

$$\frac{1}{g} = 1 + \frac{ak}{h}(1 - e^{-i\xi h}) + \frac{4k}{h^2}\sin^2\frac{\xi h}{2}$$

$$= 1 + \frac{ak}{h}(1 - \cos \xi h) + \frac{4k}{h^2}\sin^2\frac{\xi h}{2} + i\frac{ak}{h}\sin \xi h$$

$$= 1 + \frac{2ak}{h}\sin^2\frac{\xi h}{2} + \frac{4k}{h^2}\sin^2\frac{\xi h}{2} + i\frac{ak}{h}\sin \xi h$$

The real part exceeds 1. So the scheme is unconditionally stable.
Without calculation, it can be seen that the scheme is first-order in space and time. Indeed, it is essentially Forward Euler in time. Furthermore, the discretisation of the au_x term is only $O(h^2)$.