Sheet 3. Solutions

(1) We can write the system in vector form:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ u \end{pmatrix} + \underbrace{\begin{pmatrix} u & h \\ g & u \end{pmatrix}}_{A} \frac{\partial}{\partial x}\begin{pmatrix} h \\ u \end{pmatrix} = 0$$

The eigenvalues of A are found by solving

$$0 = \begin{vmatrix} u-\lambda & h \\ g & u-\lambda \end{vmatrix} = (\lambda - u)^2 - gh$$

Hence
$$\lambda_{\pm} = u \pm \sqrt{gh}$$

The eigenvalues are real and distinct as long as $h > 0$. There are two characteristic curve

$$x = X_+(t) \quad \text{and} \quad x = X_-(t)$$

that satisfy

$$\frac{dX_{\pm}}{dt} = u(X_{\pm}, t) \pm \sqrt{g\,h(X_{\pm}, t)}$$

(2) Put

$$L_j^n := u(x_j, t_{n+1}) - c_{-1}\,u(x_{j-1}, t_n) - c_0\,u(x_j, t_n) - c_1\,u(x_{j+1}, t_n)$$

and expand at (x_j, t_n). The result is

$$L_j^h = (1 - c_{-1} - c_0 - c_1)\,u(x_j, t_n) + [h(c_{-1} - c_1) - ka]\,u_x(x_j, t_n)$$
$$+ \left[\frac{k^2a^2}{2} - \frac{h^2}{2}(c_1 + c_{-1})\right] u_{xx}(x_j, t_n) + O(k^3 + h^3)$$

where we have used the fact that $u_t(x,t) = -a u_x(x,t)$. So we require

$$c_1 + c_0 + c_{-1} = 1, \quad c_{-1} - c_1 = \frac{ak}{h}, \quad c_1 + c_{-1} = \frac{a^2k^2}{h^2}.$$

The soln is $c_{-1} = \frac{ak}{2h}\left(1 + \frac{ak}{h}\right)$, $c_0 = 1 - \frac{a^2k^2}{h^2}$, $c_1 = \frac{ak}{2h}\left(\frac{ak}{h} - 1\right)$

The resulting scheme is

$$u_j^{n+1} = u_j^n - \frac{ak}{2h}\left\{ \left(1-\frac{ak}{h}\right) u_{j+1}^n + \frac{2ak}{h} u_j^n - \left(1+\frac{ak}{h}\right) u_{j+1}^n \right\}$$

If $\mu = \frac{ak}{h}$ is held fixed as $k \to 0$, then the scheme is second-order.

von Neumann analysis: Put $u_j^n = g^n e^{i\xi x_j}$, $\xi h \in [0, 2\pi]$. Then

$$g = 1 - \frac{\mu}{2}\left\{ (1-\mu) e^{i\xi h} + 2\mu - (1+\mu) e^{-i\xi h} \right\}$$

That is

$$g = 1 - 2\mu^2 \sin^2 \frac{\xi h}{2} - i\mu \sin \xi h$$

This gives

$$|g|^2 = 1 - 4\mu^2 \sin^2 \frac{\xi h}{2} + 4\mu^4 \sin^4 \frac{\xi h}{2} + 4\mu^2 \sin^2 \frac{\xi h}{2} \cos^2 \frac{\xi h}{2}$$

$$= 1 - 4\mu^2 \sin^4 \frac{\xi h}{2} + 4\mu^4 \sin^4 \frac{\xi h}{2}$$

and so the scheme is stable if $\mu \le 1$.

(3) The characteristic curve through (x_j, t_{n+1}) is the curve of eqn

$$x = x_j + a(t_{n+1} - t)$$

The CFL condition requires that the characteristic crosses the horizontal line $t = t_n$ between x_k and x_{k+1}, i.e.

$$x_{k-1} \le x_j - ah \le x_k$$

That is

$$x_j - ak + \theta h - h \le x_j - ah \le x_j - ak$$

This yields

$$0 \le \frac{ak}{h} + 1$$

and so there is no restriction.

von Neumann analyses: Put $u_j^n = g^n e^{i\xi jh}$, $\xi h \in [0, 2\pi]$. Then, after simplification,

$$g = e^{-ia\xi k + i\theta\xi h}\left[(1-\theta) + \theta e^{-i\xi h}\right]$$

It is then readily seen that $|g| \le 1$ iff $\theta + (1-\theta)\cos\xi h \le 1$.
So the scheme is unconditionally stable.

To study the consistency, we put

$$L_j^h := u(x_j, t_{n+1}) - (1-\theta)\, u(x_k, t_n) - \theta\, u(x_{k-1}, t_n)$$

and expand at (x_j, t_n). We find

$$\begin{aligned} L_j^h = {} & \cancel{u(x_j, t_n)} + k u_t(x_j,t_n) + \frac{k^2}{2} u_{tt}(x_j,t_n) + O(k^3) \\ & - (1-\theta)\Big[\cancel{u(x_j,t_n)} + (x_k - x_j)\, u_x(x_j,t_n) + \frac{(x_k-x_j)^2}{2} u_{xx}(x_j,t_n) + O\big([x_k - x_j]^3\big) \Big] \\ & - \theta \Big[\cancel{u(x_j,t_n)} + (x_{k-1}-x_j)\, u_x(x_j,t_n) + \frac{(x_{k-1}-x_j)^2}{2} u_{xx}(x_j,t_n) + O\big([x_{k-1}-x_j]^3\big) \Big] \end{aligned}$$

We use $u_t(x_j,t_n) = -a u_x(x_j,t_n)$, $x_k - x_j = \theta h - ak$ and $x_{k-1} - x_j = (\theta-1)h - ak$.
Then

$$\begin{aligned} L_j^h = {} & -\big[\, ak + (1-\theta)(\theta h - ak) + \theta(\theta h - ak - h) \,\big]\, u_x(x_j,t_n) \\ & + \big[\, a^2k^2 - (1-\theta)(\theta h - ak)^2 - \theta(\theta h - ak - h)^2 \,\big] \frac{u_{xx}(x_j,t_n)}{2} \\ & + O(k^3 + h^3) \end{aligned}$$

After simplification, we find

$$L_j^h = \big[\, a^2k^2 - (1-2\theta)(\theta h - ak)^2 + 2\theta h(\theta h - ak) - \theta h^2 \,\big] \frac{u_{xx}(x_j,t_n)}{2}$$

The scheme is first-order accurate (if $\frac{ak}{h}$ is held fixed as $k \to 0$).

(4) (a) The eqn may be expressed as

$$\frac{\partial}{\partial t}\begin{pmatrix} u_x \\ u_t \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & c^2 \\ 1 & 0 \end{pmatrix}}_{A} \frac{\partial}{\partial x}\begin{pmatrix} u_x \\ u_t \end{pmatrix}, \quad c^2 = (1+4x)^2$$

The eigenvalues of A are

$$\lambda_\pm = \pm c = \pm (1+4x)$$

So the characteristics are the curves of eqn

$$\frac{dX_\pm}{dt} = \pm (1 + 4X_\pm)$$

The characteristics through the point (x_0, t_0) are thus

$$x = X_{\pm}(t) = (x_0 + 1/4)\, e^{\pm 4(t - t_0)} - 1/4 .$$

(c) The obvious scheme is

$$u_j^{n+1} - 2u_j^n + u_j^{n-1} = (1 + 4x_j)^2 \frac{k^2}{h^2} \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right) \qquad (S)$$

For the spatial grid $0 = x_0 < x_1 < \ldots < x_N = 1$ the various boundary and initial conditions may be implemented as follows

$$u_j^0 = x_j^2 , \quad 0 \le j \le N, \qquad u_j^1 = u_j^0 , \quad 0 \le j \le N .$$

$$u_1^n = u_0^n , \quad n = 0, 1, \ldots \quad \text{and} \quad u_N^n = 1 , \quad n = 0, 1, \ldots$$

The scheme (S) is used for $n = 1, 2, \ldots$ and $j = 1, 2, \ldots, N-1$.

The characteristics through (x_j, t_{n+1}) are the curves of eqn

$$x = (x_j + 1/4)\, e^{4(t - t_{n+1})} - 1/4 \quad \text{and} \quad x = (x_j + 1/4)\, e^{-4(t - t_{n+1})} - 1/4$$

The CFL condition is that they should cross the line $t = t_n$ between x_{j-1} and x_{j+1}. This gives

$$x_{j-1} + 1/4 \le (x_j + 1/4)\, e^{-4k} \quad \text{and} \quad (x_j + 1/4)\, e^{4k} \le x_{j+1} + 1/4$$

It is readily seen that, for small k, this reduces to

$$(4x_j + 1) \frac{k}{h} \le 1$$

Since $x \in (0,1)$, a necessary condition for stability is therefore

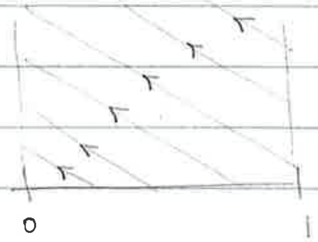
$$5 \frac{k}{h} \le 1 .$$

(5) The characteristic curve through (x_j, t_{n+1}) is

$$x = x_j + (t_{n+1} - t)$$

For $j=0$, the curve crosses the $t = t_n$ axis at $x = k$. For $\mu = 0.9$, the CFL condition is satisfied if we use

$$u_0^{n+1} = u_1^n$$



So this should work well. In view of the geometry of the characteristics for this problem, the other option bears no relation to the exact soln.

(6)(a) Put $u_j^n = g^n e^{i\xi x_j}$, $\xi h \in [0, 2\pi]$. Then

$$g^{n+1} e^{i\xi x_j} = g^n e^{i\xi x_j} - \frac{\mu}{2} g^{n+1} e^{i\xi x_j} \left(e^{i\xi h} - e^{-i\xi h} \right)$$

After simplification, this gives

$$(1 + i\mu \sin \xi h) g = 1$$

Obviously, $|g|^2 = \dfrac{1}{1 + \mu^2 \sin^2 \xi h} \leq 1 \quad \forall \ \xi h \in [0, 2\pi]$

(b) Put $g = e^{i\omega k}$ in (a). Then

$$\omega = \frac{1}{ik} \ln \frac{1}{1 + i\mu \sin \xi h} = -\frac{1}{ik} \ln (1 + i\mu \sin \xi h)$$

We use $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \ldots$ Then

$$\omega = -\frac{1}{ik} \left\{ i\mu \sin \xi h + \frac{\mu^2}{2} \sin^2 \xi h - \frac{i\mu^3}{3} \sin^3 \xi h + \ldots \right\}$$

Writing $\omega = \omega_r + i\omega_i$, we find, for small wave numbers,

$$\omega_r = -a\xi \frac{\sin \xi h}{\xi h} + k^2 \frac{a^3}{3} \xi^3 \frac{\sin^3 \xi h}{\xi^3 h^3} + \ldots$$

$$\omega_i = i\frac{k}{2} a^2 \xi^2 \frac{\sin^2 \xi h}{\xi^2 h^2} + \ldots$$

Using $\frac{\sin z}{z} = 1 - \frac{z^2}{6} + O(z^4)$, we deduce

$$\omega_r = -a\xi \left\{ 1 - \frac{\xi^2 h^2}{6} + \frac{k^2 a^3}{3}\xi^3 + \cdots \right\}$$

and

$$\omega_i = \frac{ik}{2} a^2 \xi^2 + k\,O(\xi^2 h^2)$$

For small wave numbers, the dispersive error is, to leading order,

$$\omega_r + a\xi = \frac{a}{6}\xi^3 h^2 \quad \text{as } \xi \to 0$$

The scheme is dissipative, i.e. the amplitude of the plane wave solution decays with n.

(7) (a) Putting $u_j^n = e^{i[\omega t_n + \xi x_j]}$, we obtain

$$e^{i\omega k} = 1 + \frac{\mu}{2}\left(e^{i\xi h} - e^{-i\xi h}\right) + \frac{\mu^2}{2}\left(e^{i\xi h} - 2 + e^{-i\xi h}\right)$$

$$= 1 - 2\mu^2 \sin^2\frac{\xi h}{2} + i\mu \sin \xi h$$

Hence

$$ik\omega = \ln\left(1 + i\mu\sin\xi h - 2\mu^2\sin^2\frac{\xi h}{2}\right)$$

$$= i\mu\sin\xi h - 2\mu^2\sin^2\frac{\xi h}{2} - \frac{1}{2}\left[i\mu\sin\xi h - 2\mu^2\sin^2\frac{\xi h}{2}\right]^2 + \cdots$$

$$= i\mu\sin\xi h - 2\mu^2\sin^2\frac{\xi h}{2} + \frac{\mu^2}{2}\sin^2\xi h + 2i\mu^3\sin\xi h\sin^2\frac{\xi h}{2}$$
$$- 2\mu^4\sin^4\frac{\xi h}{2} + \cdots$$

$$= i\mu\sin\xi h - 2\mu^2\sin^2\frac{\xi h}{2}\left[1 - \cos^2\frac{\xi h}{2}\right] + 2i\mu^3\sin\xi h\sin^2\frac{\xi h}{2}$$
$$- 2\mu^4\sin^4\frac{\xi h}{2}$$

$$= i\mu\sin\xi h + 2\mu^2\sin^4\frac{\xi h}{2} + 2i\mu^3\sin\xi h\sin^2\frac{\xi h}{2} - 2\mu^4\sin^4\frac{\xi h}{2}$$
$$+ \cdots$$

We deduce

$$\omega_r = \xi \frac{\sin \xi h}{\xi h} + 2k^2 \xi^3 \frac{\sin \xi h}{\xi h} \frac{\sin^2 \xi h/2}{\xi^2 h^2} + \cdots$$

$$= \xi \left[1 - \frac{\xi^2 h^2}{2} + \cdots \right] + \frac{1}{2} k^2 \xi^3 \left[1 + O(\xi^2 h^2) \right]$$

The dispersion error is

$$\omega_r - \xi = \frac{\xi^3}{2} (\mu - 1) h^2 + O(\xi^5)$$

There is a very small amount of (anti) dissipation:

$$\omega_i = - \frac{2k}{h^2} \sin^4 \frac{\xi h}{2} + \cdots = - \frac{k}{8} h^2 \xi^4 + \cdots$$

(b) Put $u_j^n = e^{i(\omega t_n + \xi x_j)}$. Then

$$e^{i\omega k} = 1 + \mu (e^{i\xi h} - 1) = 1 - \mu (1 - \cos \xi h) + i\mu \sin \xi h$$

$$= 1 - 2\mu \sin^2 \frac{\xi h}{2} + i\mu \sin \xi h$$

Hence

$$ik\omega = \ln \left(1 - 2\mu \sin^2 \frac{\xi h}{2} + i\mu \sin \xi h \right)$$

$$= -2\mu \sin^2 \frac{\xi h}{2} + i\mu \sin \xi h - \frac{1}{2} \left(-2\mu \sin^2 \frac{\xi h}{2} + i\mu \sin \xi h \right)^2 + \cdots$$

$$= -2\mu \sin^2 \frac{\xi h}{2} - 2\mu^2 \sin^4 \frac{\xi h}{2} + \frac{1}{2} \mu^2 \sin^2 \xi h$$

$$+ i\mu \sin \xi h + 2i\mu^2 \sin \xi h \sin^2 \frac{\xi h}{2} + \cdots$$

We deduce

$$\omega_r = \xi \frac{\sin \xi h}{\xi h} + \frac{k}{2} \xi^3 \frac{\sin \xi h}{\xi h} \frac{\sin^2 \frac{\xi h}{2}}{(\xi h/2)^2} + \cdots$$

$$= \xi \left[1 - \frac{\xi^2 h^2}{6} + \frac{k}{2} \xi^3 + \cdots \right]$$

and

$$k\,\omega_i = 2\mu \sin^2 \frac{\xi h}{2} - \frac{1}{2}\mu^2 \sin^2 \xi h + 2\mu^2 \sin^4 \frac{\xi h}{2} + \cdots$$

$$= 2\mu \sin^2 \frac{\xi h}{2} \left(1 - \mu \cos^2 \frac{\xi h}{2}\right) + 2\mu^2 \sin^4 \frac{\xi h}{2} + \cdots$$

$$= 2\mu(1-\mu) \frac{\xi^2 h^2}{4} + O(\xi^4 h^4)$$

That is

$$\omega_i = \frac{h}{2}(1-\mu)\,\xi^2 + \cdots$$

(c) We have

$$e^{ik\omega} = e^{-ik\omega} + \mu\left[e^{i\xi h} - e^{-i\xi h}\right]$$

Hence

$$\sin(k\omega) = \mu \sin \xi h$$

Put $z = \mu \sin \xi h$ and recall that $\arcsin z = z + \frac{z^3}{6} + \cdots$

Then

$$k\omega = \mu \sin \xi h + \frac{\mu^3}{6} \sin^3 \xi h + \cdots$$

The scheme is non-dissipative and

$$\omega = \xi \frac{\sin \xi h}{\xi h} + \frac{k^2}{6} \xi^3 \frac{\sin^3 \xi h}{\xi^3 h^3} + \cdots$$

$$= \xi \left\{ 1 - \frac{\xi^2 h^2}{6} + \frac{k^2}{6} \xi^2 + O(\xi^4) \right\}$$

$$= \xi \left\{ 1 + (\mu - 1) \frac{h^2}{6} \xi^2 + O(\xi^4) \right\}$$