Solution Problem (7)

(a) We have $T_n = \cos n\theta$, where $x = \cos\theta$

Then $T_{2n}(x) = T_n(2x^2-1)$ amounts to

$$\underbrace{T_{2n}(\cos\theta)}_{\cos 2n\theta} = \underbrace{T_n(\underbrace{2\cos^2\theta - 1}_{\cos 2\theta})}_{\cos 2n\theta}$$

On the other hand, $\cos 2n\theta = 2\cos^2 n\theta - 1$,

and so $T_{2n}(x) = 2T_n^2(x) - 1$

(b) We use the identity

$$\underbrace{\cos(n+1)\theta}_{T_{n+1}} + \underbrace{\cos(n-1)\theta}_{T_{n-1}} = 2\underbrace{\cos\theta}_{x}\,\underbrace{\cos n\theta}_{T_n}$$

(c) $$\frac{dT_n}{dx} = \frac{d\theta}{dx}\frac{d}{d\theta}\cos n\theta = \frac{n\sin n\theta}{\sin\theta}$$

$$\left(\frac{dx}{d\theta}\right)^{-1} = -\frac{1}{\sin\theta}$$

(7)

(d) $$\frac{d^2 T_n}{dx^2} = -\frac{1}{\sin\theta}\frac{d}{d\theta}\frac{n\sin n\theta}{\sin\theta}$$

$$= -\frac{1}{\sin\theta}\left(\frac{n^2\cos n\theta}{\sin\theta} - \frac{n\cos\theta\sin n\theta}{\sin^2\theta}\right)$$

Solution Problem 8

(a) Let $u(x) = \sum_{n=0}^{N-1} a_n T_n(x) \Big\}$ N unknowns

and require

$$< T_m, u_{xx} + \lambda u > = 0 \quad \text{for } m = 0, \dots, N-3.$$

Together with $u(1) = u(-1) = 0$ N equations.

This yields

$$N \text{ equ.} \begin{cases} \sum_{n=0}^{N-1} a_n < T_m, T_n'' + \lambda T_n > = 0, & m = 0, \dots, N-3 \\ \sum_{n=0}^{N-1} a_n T_n(\pm 1) = 0 \end{cases}$$

In addition, $< T_m, T_n > = \delta_{mn} \frac{\pi}{2} (\delta_{n0} + 1)$

$T_n(1) = 1, \quad T_n(-1) = (-1)^n$

Thus we obtain a matrix

$$
M = \begin{bmatrix}
\langle T_0, T_0'' \rangle + \lambda\pi & \langle T_0, T_1'' \rangle & \cdots & \langle T_0, T_{N-1}'' \rangle \\
\langle T_1, T_0'' \rangle & \langle T_1, T_1'' \rangle + \lambda \frac{\pi}{2} & \cdots & \langle T_1, T_{N-1}'' \rangle \\
\vdots & \vdots & & \vdots \\
\langle T_{N-3}, T_0'' \rangle & & & \langle T_{N-3}, T_{N-1}'' \rangle \\
1 & -1 & & (-1)^{N-1} \\
1 & 1 & & 1
\end{bmatrix}
$$

Need to find λ such that $\det(M) = 0$.

(b) Again $u(x) = \sum_{n=0}^{N-1} a_n T_n(x)$,

but now

$$\sum_{n=0}^{N-1} a_n T_n(-1) = 0 \qquad \text{eqn } j = 1$$

$$\sum_{n=0}^{N-1} a_n \left(T_n''(x_j) + \lambda T_n(x_j) \right) = 0 \qquad j = 2, \ldots, N-1$$

$$\sum_{n=0}^{N-1} a_n T_n(1) = 0 \qquad j = N$$

where $x_j = \cos\left[\frac{\pi(2j-1)}{2N}\right]$ collocation points

Thus we obtain the matrix

$$
M = \begin{bmatrix}
1 & -1 & \cdots & (-1)^{N-1} \\
T_0''(x_2) + \lambda T_0(x_2) & T_1''(x_2) + \lambda T_1(x_2) & \cdots & T_{N-1}''(x_2) + \lambda T_{N-1}(x_2) \\
T_0''(x_3) + \lambda T_0(x_3) & T_1''(x_3) + \lambda T_1(x_3) & \cdots & T_{N-1}''(x_3) + \lambda T_{N-1}(x_3) \\
\vdots & \vdots & & \vdots \\
T_0''(x_{N-1}) + \lambda T_0(x_{N-1}) & T_1''(x_{N-1}) + \lambda T_1(x_{N-1}) & \cdots & T_{N-1}''(x_{N-1}) + \lambda T_{N-1}(x_{N-1}) \\
1 & 1 & & 1
\end{bmatrix}
$$

and again we require $\det(M) = 0$

<u>Remark</u>: the general form of (a) and (b) is that of a <u>generalized eigenvalue problem</u>. $\boxed{\underline{A}\,\underline{x} = \lambda\,\underline{B}\,\underline{x}}$ There are efficient numerical routines e.g. LAPACK

(c) If the boundary condition is built in, the structure simplifies.

The system of equations is now

$$\sum_{n=0}^{N-1} a_n \langle \phi_m, \phi_n'' + \lambda \phi_n \rangle = 0,$$

which corresponds to

$$\begin{bmatrix} \langle \phi_0, \phi_0'' \rangle & & \langle \phi_0, \phi_{N-1}'' \rangle \\ & & \\ \langle \phi_{N-1}, \phi_0'' \rangle & & \langle \phi_{N-1}, \phi_{N-1}'' \rangle \end{bmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \end{pmatrix} = -\lambda \begin{bmatrix} \langle \phi_0, \phi_0 \rangle & \langle \phi_0 \phi_{N-1} \rangle \\ & \\ \langle \phi_{N-1}, \phi_0 \rangle & \langle \phi_{N-1} \phi_{N-1} \rangle \end{bmatrix} \begin{pmatrix} a_0 \\ \\ a_{N-1} \end{pmatrix}$$

In the case $\phi_n = T_{n+2} - T_n$

$$\langle \phi_m, \phi_n \rangle = \langle T_{m+2} - T_m, T_{n+2} - T_n \rangle =$$

$$\delta_{mn} \frac{\pi}{2} + \delta_{mn} \frac{\pi}{2}(\delta_{n0} + 1) - \delta_{m,n+2} \frac{\pi}{2} - \delta_{m+2,n} \frac{\pi}{2}$$

$$B = -\begin{bmatrix} 3\pi/2 & 0 & -\pi/2 & & & & 0 \\ 0 & \pi & & \ddots & & & \\ -\pi/2 & & \ddots & & \ddots & & \\ & \ddots & & \ddots & & \ddots & \\ & & \ddots & & \ddots & & -\pi/2 \\ 0 & & & \ddots & & \ddots & 0 \\ & & & & -\pi/2 & 0 & \pi \end{bmatrix}$$

which can easily be inverted to produce a regular eigenvalue problem.

Other cases are treated similarly!