Solution Problem (1)

$u_{xx} = f(x) \quad u(0) = 0 \ \& \ u(1) = 1$

let $u_j = u(x_j) \quad x_j = j/N \quad j = 1, \dots N-1$

use 2nd order centred differences

$$u_{j+1} - 2u_j + u_{j-1} = h^2 f(x_j) \quad (*) . \quad j = 1, \dots N-1 \text{ with } \left.\begin{array}{l} u_0 = 0 \\ \& \ u_N = 1 \end{array}\right\} \text{B.C.s}$$

soln for $f(x) = 2$ is just $u = x^2$

so need to check $u_j = \left(\frac{j}{N}\right)^2$ solves $(*)$

$$\text{LHS} = \frac{(j+1)^2}{N^2} - 2\frac{j^2}{N^2} + \frac{(j-1)^2}{N^2} = \frac{(\cancel{j^2} + \cancel{2j} + 1) - \cancel{2j^2} + (\cancel{j^2} - \cancel{2j} + 1)}{N^2} = \frac{2}{N^2} = 2h^2$$
$$= \text{RHS}$$

So discretized soln is the exact soln !

This is because the truncation introduced by the 2nd order difference approximation vanishes for this simple soln (i.e. $\frac{d^4u}{dx^4} \equiv 0 \ \forall x$)

This will also be true for other cases of f provided $\frac{d^4u}{dx^4} = 0$

$\Rightarrow$ u is a cubic so f must be a linear polynomial $= Ax + B$

In 2D $\quad u_{xx} + u_{yy} = f(x,y)$ need both $\frac{\partial^4 u}{\partial x^4}$ & $\frac{\partial^4 u}{\partial y^4}$ to vanish

hence most general soln can be up to a cubic in each of x & y

$$u = a(x)y^3 + b(x)y^2 + c(x)y + d(x)$$ where a, b, c & d are cubic polynomials in x

hence most general f is

$$f = \nabla^2 \left\{ a(x)y^3 + b(x)y^2 + c(x)y + d(x) \right\}$$

[So e.g. $\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$ must differentiate $u = x^3$ exactly

check $$\frac{(x+h)^3 - 2x^3 + (x-h)^3}{h^2} = \frac{\cancel{x^3} + \cancel{3x^2h} + 3xh^2 + \cancel{h^3} - \cancel{2x^3} + (\cancel{x^3} - \cancel{3x^2h} + 3xh^2 - \cancel{h^3})}{h^2}$$
$$= 6x \quad \checkmark \quad]$$

Solution Problem (2)

$$\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \underline{x} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

<u>Conjugate Gradient Method</u>

$$\underline{x}_0 = \underline{0} \; ; \; \underline{r}_0 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \; \& \; p_0 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\alpha_1 = \frac{r_0^T r_0}{p_0^T A p_0} = \frac{(2\ 0\ 0)\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}{(2\ 0\ 0)\underbrace{\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}}} = \frac{1}{3}$$

$$\underline{x}_1 = \underline{x}_0 + \frac{1}{3}\underline{p}_0 = \begin{pmatrix} 2/3 \\ 0 \\ 0 \end{pmatrix}$$

$$\underline{r}_1 = \underline{r}_0 - \alpha_1 A p_0 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\left(\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ -2/3 \end{pmatrix}$$

$$\beta_1 = \frac{\underline{r}_1^T . \underline{r}_1}{\underline{r}_0^T . \underline{r}_0} = \frac{4/9}{4} = \frac{1}{9}$$

$$\underline{p}_1 = \underline{r}_1 + \beta_1 \underline{p}_0 = \begin{pmatrix} 0 \\ 0 \\ -2/3 \end{pmatrix} + \frac{1}{9}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2/9 \\ 0 \\ -2/3 \end{pmatrix}$$

<u>repeat</u>

$$\alpha_2 = \frac{\underline{r}_1^T . \underline{r}_1}{p_1^T A p_1} = \frac{4/9}{(2/9\ \ 0\ \ -2/3)\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2/9 \\ 0 \\ -2/3 \end{pmatrix}} = \frac{4/9}{8/27} = \frac{3}{2}$$

$$\underline{x}_2 = \underline{x}_1 + \frac{3}{2}\underline{p}_1 = \begin{pmatrix} 2/3 \\ 0 \\ 0 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 2/9 \\ 0 \\ -2/3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\underline{r}_2 = \underline{r}_1 - \alpha_2 A \underline{p}_1 = \begin{pmatrix} 0 \\ 0 \\ -2/3 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2/9 \\ 0 \\ -2/3 \end{pmatrix}$$

(residual)

$$= \begin{pmatrix} 0 \\ 0 \\ -2/3 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} 0 \\ 0 \\ -4/9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ so we've reached sol!!

$$\Rightarrow \text{sol}^{\underline{n}} \text{ is } \underline{x} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Now $A^3 - 6A^2 + 10A - 4I = 0$

apply A^{-1} $\quad \frac{1}{4}(A^2 - 6A + 10I) = A^{-1}$

$\therefore$ if $A\underline{x} = \underline{b} \Rightarrow \underline{x} = \frac{1}{4}(A^2 - 6A + 10I)\underline{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ check.