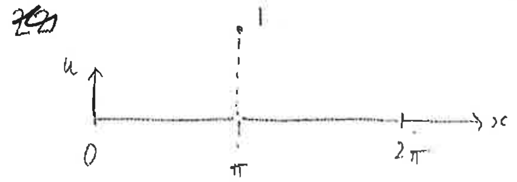
# Solution Problem (3)




$$u_t = u_{xx} \qquad (*)$$

let $u(x,t) = \sum_{k}^{-K} \hat{u}_k^{(t)} e^{ikx}$

$$u_t = \sum_{k}^{-K} \frac{d\hat{u}_k}{dt} e^{ikx} \quad \& \quad u_{xx} = \sum_{k}^{-K} -k^2 \hat{u}_k e^{ikx}$$

So $$\frac{d\hat{u}_k}{dt} = -k^2 \hat{u}_k \implies \hat{u}_k = \hat{u}_k(0) e^{-k^2 t}$$

initial transform $\hat{u}_k(0)$?

perform $$\hat{u}_k(0) = \frac{1}{N} \sum_{l=1}^{N} u(x_l) e^{-ikx_l} \qquad \text{where } x_l = \frac{2\pi l}{N}$$

$$= \frac{1}{N} 1 . e^{-ik\pi}$$ (where we have assumed $\exists$ a grid pt at $\pi$ so $u(x_l) = \begin{cases} 1 & \text{at } x_l = \pi \\ 0 & \text{otherwise} \end{cases}$ )

$$\implies \hat{u}_k(t) = \frac{1}{N} e^{-ik\pi - k^2 t}$$

$$\therefore \quad u(x_j, t) = \sum_{k=K}^{K} \frac{1}{N} e^{-ik\pi - k^2 t + ikx_j}$$

$$= \frac{1}{N} \sum_{k=-K}^{K} e^{ik(x_j - \pi) - k^2 t} \qquad \text{which is } (*)$$

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RTS confirm $u(x,t) = \frac{1}{N} \sum_{-K}^{K} e^{ik(x-\pi) - k^2 t}$ satisfies heat equation.

$$u_t = \frac{1}{N} \sum_{-K}^{K} -k^2 e^{ik(x-\pi) - k^2 t} = u_{xx} = \sum_{K}^{K} (ik)^2 e^{ik(x-\pi) - k^2 t}$$

(Check)

The i.c. has transform $\hat{u}_k(0) = \frac{1}{N} e^{-ik\pi} = O(1) \ \forall k$ so no drop off.

(as expected since $u(x,0)$ is not even $C^0$)

Instead try $u(x,0) = \cos mx$ (which can be made entire across $\mathbb{C}$ plane)

then $$\hat{u}_k(0) = \begin{cases} 0 & |k| \neq m \\ \frac{1}{2} & |k| = m \end{cases}$$ so drop off is dramatic!

($\hat{u}_k$ has compact support)

# Solution Problem (4)

We have $u^{(p)}(x_\ell) =: u_\ell^{(p)} = D_{\ell q}^{(p)}(x_\ell - x_q)\, u_q$,

where the differentiation operator is

$$D_{\ell q}^{(p)}(\varrho) = \frac{1}{N} \sum_{k=-K}^{K} \frac{d^p}{d\varrho^p} e^{ik\varrho} = \frac{1}{N} \frac{d^p}{d\varrho^p} \left( \frac{\sin N\varrho/2}{\sin \varrho/2} \right).$$

Then the spatial discretization is

$[u u_x]_\ell = u_\ell D_{\ell q}^{(1)} u_q$ and

$[u_{xx}]_\ell = D_{\ell q}^{(2)} u_q$, so that the Crank-Nicolson scheme becomes

$$\boxed{\frac{u_\ell^{n+1} - u_\ell^n}{\Delta t} + \frac{1}{2}\left( u_\ell^n D_{\ell q}^{(1)} u_q^n + u_\ell^{n+1} D_{\ell q}^{(1)} u_q^{n+1} \right) = D_{\ell q}^{(2)} \left( \frac{u_q^{n+1} + u_q^n}{2} \right)}$$

<u>von Neumann analysis</u>:

$u_\ell^n = g^n e^{ikx_\ell}$, and so

$u_\ell^p = (ik)^p g^n e^{ikx_\ell}$, while

$u_\ell^n = \bar{u}$, so that

$$\boxed{\frac{g-1}{\Delta t} + \bar{u}\, ik\, \frac{g+1}{2} = -\frac{g+1}{2} k^2}$$

This is of the form $g-1 = -\frac{a}{2}(g+1)$,

with $a = \Delta t\,(k^2 + ik\bar{u})$.

Solving for $g$: $g = \frac{2-a}{2+a}$; stable

for $|2-a| \leq |2+a|$, satisfied for all

$\mathrm{Re}\{a\} \geq 0$. Thus the scheme is

unconditionally stable.

Order of the scheme:

In the time domain, I can write

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{f^{n+1} + f^n}{2}, \text{ where}$$

$$f^n = -u_e^n D_{eq}^{(1)} u_q^n + D_{eq}^{(2)} u_q^n$$

Now

$$u^{n/n+1} = u^{n+1/2} \mp \frac{\Delta t}{2} u_t^{n+1/2} + \frac{1}{2}\left(\frac{\Delta t}{2}\right)^2 u_{tt}^{n+1/2} + \dots$$

and the same for $f^{n/n+1}$.

Thus $\dfrac{u^{n+1} - u^n}{\Delta t} = u_t^{n+1/2} + O(\Delta t)^2$ and

$$\frac{f^{n+1} + f^n}{2} = f^{n+1/2} + \left(\frac{\Delta t}{2}\right)^2 f_{tt}^{n+1/2}$$

which mean that

$u_t^{n+1/2} = f^{n+1/2} + O(\Delta t)^2$, i.e. the scheme is of 2nd order in time.

# Solution Problem (5)

~~1~~. $u(x) = \cos(mx)$

obviously $u u_x = \cos(mx)\left[-m\sin(mx)\right] = -m\cos(mx)\sin(mx) = -\frac{m}{2}\sin(2mx)$

Now attempt convolution approach

$$\left(\widehat{u u_x}\right)_k = \sum_{\substack{q=-K \\ p+q=k}}^{K} iq\,\hat{v}_p\,\hat{u}_q \qquad \text{where} \quad \hat{u}_k = \begin{cases} \frac{1}{2} & |k| = m \\ 0 & |k| \neq m \end{cases} \quad \Bigg|\ \text{FT of } \cos mx$$

$$= \sum_{q=-K}^{K} iq\,\hat{u}_q\,\hat{v}_{k-q}$$

$$= i(-m)\,\hat{u}_{-m}\,\hat{u}_{k+m} + i(+m)\,\hat{u}_m\,\hat{v}_{k-m}$$

$$= -im\left(\tfrac{1}{2}\right)\tfrac{1}{2}\,\delta_{m\,|k+m|} + im\left(\tfrac{1}{2}\right)\tfrac{1}{2}\,\delta_{m\,|k-m|}$$

only non zero $k$ are

$k = 0 \qquad \left(\widehat{v v_x}\right)_0 = -im\,\frac{1}{4} + im\,\frac{1}{4} = 0$

$k = 2m \qquad \left(\widehat{v v_x}\right)_{2m} = \frac{im}{4}$

$k = -2m \qquad \left(\widehat{v v_x}\right)_{-2m} = -\frac{im}{4}$

so $$\left(\widehat{u u_x}\right)_k = \begin{cases} 0 & |k| \neq 2m \\ im/4 & k = 2m \\ -im/4 & k = -2m \end{cases}$$

now $FT^{-1}$ to get $\widehat{v v_x} = \frac{im}{4}e^{2imx} - \frac{im}{4}e^{-2imx} = -\frac{m}{2}\sin 2mx$

as required

Spectral Space $\longleftarrow$ | $\longrightarrow$ Physical Space

In practice

$$\hat{u}_k = \begin{cases} \frac{1}{2} & |k| = m \\ 0 & |k| \neq m \end{cases} \xrightarrow{FT^{-1}} \cos mx$$

$$ik\hat{u}_k = \begin{cases} \frac{1}{2}ik & |k| = m \\ 0 & |k| \neq m \end{cases} \xrightarrow{FT^{-1}} -m\sin(mx)$$

$$\left.\begin{matrix}\cos mx \\ -m\sin(mx)\end{matrix}\right\} \quad u u_x = -\frac{m}{2}\sin(2mx)$$

$$\left(\widehat{v v_x}\right)_k = \begin{cases} 0 & |k| \neq 2m \\ im/4 & k = 2m \\ -im/4 & k = -2m \end{cases} \xleftarrow{FT} u u_x$$

# Solution Problem (6)

We have $F_\ell^{-1}\{\hat{u}_k\} \equiv u(x_\ell) = \sum_{p=-K}^{K} \hat{u}_p e^{ipx_\ell}$

and $F_\ell^{-1}\{ik\hat{u}_k\} = \sum_{q=-K}^{K} iq\,\hat{u}_q e^{iqx_\ell}$

and so

$$\tilde{\omega}_k = \frac{1}{N}\sum_{\ell=1}^{N} u u_x(x_\ell)\, e^{-ikx_\ell} =$$

$$\frac{1}{N}\sum_{\ell=1}^{N}\sum_{p,q=-K}^{K} \hat{u}_p\, iq\, \hat{u}_q\, e^{i(p+q-k)x_\ell}$$

Now $\frac{1}{N}\sum_{\ell=1}^{N} e^{ipx_\ell} = \sum_{m=-\infty}^{\infty} \delta_{p,mN}$

For $-K \le k \le K$, $|p+q-k| < 2N = 4K+2$,

and so $p+q-k = \begin{cases} 0 \\ \pm N \end{cases}$.

This means that

$$\boxed{\tilde{\omega}_k - \hat{\omega}_k = \sum_{\substack{p,q=-K \\ p+q=\pm N+k}}^{K} \hat{u}_p\, iq\, \hat{u}_q}$$

Can write this also in the form

$$\tilde{\omega}_k - \hat{\omega}_k = \sum_{p=-K+k-1}^{-K} \hat{u}_p\, i(-2K+k-1-p)\, \hat{u}_{-2K+k-1-p}$$

for $k > 0$ and

$$\tilde{\omega}_k - \hat{\omega}_k = \sum_{p=K+k+1}^{K} \hat{u}_p\, i q\, u_q \, , \quad q = 2K+1+k-p$$

for $k < 0$.

In the case $u = \cos mx = \frac{1}{2}(e^{imx} + e^{-imx})$

$$\hat{u}_p = \frac{1}{2}(\delta_{pm} + \delta_{p,-m}), \quad \hat{u}_q = \frac{1}{2}(\delta_{qm} + \delta_{q,-m})$$

and so

$$\boxed{\tilde{\omega}_k - \hat{\omega}_k = \frac{im}{4}\left[\delta_{k,-2(K-m)-1} + \delta_{k,1+2(K-m)}\right]}$$