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Stability of a viscous pinching thread

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We consider the dynamics of a fluid thread near pinch-off, in the limit that inertial effects can be neglected. There exists an infinite hierarchy of similarity solutions corresponding to pinch-off. Only one of the similarity solutions (the "ground state") is stable, all other solutions are linearly unstable to perturbations, and thus cannot be observed. Eigenvalues and eigenfunctions are calculated analytically. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4732545]

I. INTRODUCTION

It is well established that the pinch-off of a liquid thread is described by self-similar solutions of the equations of fluid motion.[1,2] Papageorgiou[3] found a similarity solution for the case that inertia can be neglected. However, sufficiently close to pinch-off this "Stokes" solution crosses over to the "Navier-Stokes" solution,[1] where surface tension, viscosity, and inertia balance.[4]

Brenner *et al.*[5] showed that both in the Stokes and the Navier-Stokes case, there exists not only one but an infinite sequence of similarity solutions. Thus, it is important to ask which member of the sequence of solutions is realized in practice. To answer this question, Brenner *et al.*[5] considered the nonlinear stability of each solution against perturbations which are localized on the scale of the base solution. They found that a perturbation of finite amplitude is necessary to destabilize it. On the basis of this calculation, they conjectured that all solutions are linearly stable, and that the solutions that were found initially[1,3] are distinguished only by the fact that the critical amplitude needed to destabilize them is greater than that for other members of the hierarchy.

Here, we focus on the Stokes case, for which similarity solutions can be found analytically[6,7] by transforming to Lagrangian variables. This approach introduces a natural ordering to the hierarchy of solutions, in that the ground state solution possesses a generic quadratic minimum. Solutions of higher order ($i = 1, 2, \ldots$, the "excited states") have a minimum of order $2(i + 1)$. Using the terminology of ground and excited states, we do not wish to imply a deep connection with quantum states in atomic physics. We simply to emphasize that there is a special state that is distinguished from all the others. In the present paper, we show that all excited states are linearly *unstable*, while the ground state alone is stable. The unstable modes have a typical wavelength comparable to that of the base solution, so our result is consistent with the short wavelength calculation of Ref. 5.

To perform the stability analysis, we follow Ref. 8 and rewrite the original equation of motion in self-similar variables. The resulting dynamical system has all the similarity solutions as *fixed points*, which facilitates the stability analysis greatly. In Ref. 9, this technique was used to study the stability of a cylinder, thinning under the action of surface diffusion. In this problem, once again an infinite hierarchy of similarity solution was found by solving the similarity equation numerically. By discretizing the problem in the spatial similarity variable, the problem was reduced to finding the eigenvalues of a large matrix. It was shown that only the ground state solution is stable, while the next few similarity solutions were unstable. By contrast, our calculation for the corresponding Stokes flow problem is performed analytically. Apart from the wide interest the fluid pinch-off problem has received,[2,6] we hope to provide more insight into the structure of the problem than is possible with a numerical approach.

Our paper is organized as follows: in Sec. II, we give the essential steps of the analytical calculation of the similarity solutions, as far as needed for the subsequent development. In Sec. III, we set up the stability calculation, leading to an ordinary differential equation the eigenfunctions have to satisfy. Using this setup, the eigenvalue spectrum for ground state and excited state solutions is calculated in Sec. IV. In Sec. V, we relate to the Eulerian description, and discuss possible extensions of the present approach.

II. SIMILARITY SOLUTIONS

We consider an axisymmetric fluid thread, driven by surface tension, in the limit that fluid inertia can be neglected. In the case of a slender jet, the equations of motion consist of two coupled equations for the thread radius and for the axial velocity field;[10] near pinch-off, this is a consistent description of the full hydrodynamic problem.[2] The equations simplify considerably if written in Lagrangian coordinates.[6] If s is a fluid particle label and $H(s, t)$ the thread radius at particle position s and time t, the equation of motion can be written as[2]

$$H_t(s,t) = \frac{\gamma}{6\eta}\left(1 - \frac{T(t)}{\gamma H(s,t)}\right), \tag{1}$$

where γ is the surface tension and η the fluid viscosity. The time-dependent parameter $T(t)$ is the total force (or tension) over the cross section of the thread; this force must be a constant along the thread.

Equation (1) would of course be trivial to solve if there were no tension, or if $T(t)$ were prescribed. However, the tension needs to be found self-consistently, making the problem non-local. It is straightforward to show[2] that

$$T(t) = \gamma\left(\int_{s_-}^{s_+}\frac{ds}{H^4}\right)^{-1}\left[\int_{s_-}^{s_+}\frac{ds}{H^3} + 3(v_+ - v_-)/v_\eta\right], \tag{2}$$

where v_-,v_+ are the boundary values of the velocity at some boundary points s_-,s_+. We will see that the integral inside the square brackets *diverges* at the pinch point, so the boundary values v_-,v_+ do not affect the pinching behavior.

We want to solve the coupled system (1) and (2) in the limit that the thread thickness H goes to zero.[6,11] We now recall briefly how the solution to the problem is found in the form of an infinite sequence of local similarity solutions. This is necessary to establish the notation, and to establish some relations needed in the later development. Breakup is assumed to occur at time t_0 and location s_0 (in Lagrangian the variable). We define a dimensionless distance to the singularity as

$$t' = \frac{\gamma(t_0 - t)}{\eta r_0}, \quad s' = \frac{s - s_0}{r_0^3}, \tag{3}$$

where r_0 is some axial reference length. To achieve a balance in (1), H and T have to go to zero linearly with t'. Thus, we will look for a similarity solution of the form

$$H = t' r_0 \overline{\chi}(\zeta), \qquad T = t'\overline{T}_0 \gamma r_0, \tag{4}$$

where the similarity variable is $\zeta = s'/t'^\delta$, and δ is the spatial similarity exponent. Inserting (4) into (1), we obtain the similarity equation

$$\overline{T}_0 = \overline{\chi} + 6\overline{\chi}^2 - 6\delta\zeta\overline{\chi}'\overline{\chi}, \tag{5}$$

where $\overline{T}_0$ is a constant.

The similarity solution also has to satisfy a condition for $\zeta \to \pm\infty$, which ensures that it matches to a time-independent outer solution. This means that

$$H_t = r_0\left[-\overline{\chi} + \delta\zeta\overline{\chi}'\right] \tag{6}$$

has to go to zero as $t' \to 0$ at a *constant value of* s'. Solving for the square bracket to be zero, we obtain

$$\overline{\chi} \propto \zeta^{1/\delta}, \quad \zeta \to \pm\infty \tag{7}$$

as a growth condition on $\overline{\chi}$. In similarity variables, the constraint (2) becomes

$$\overline{T}_0 = \left(\int_{-\infty}^{\infty} \frac{d\zeta}{\overline{\chi}^3}\right)^{-1} \int_{-\infty}^{\infty} \frac{d\zeta}{\overline{\chi}^4}. \tag{8}$$

Notice that the boundary velocity has dropped out in the limit $t' \to 0$. To summarize, we want to find a singularity-free solution of the similarity equation (5), subject to the boundary conditions (7), and satisfying the constraint (8).

Potential singularities of (5) occur at $\zeta = 0$, so we have to require a regularity condition at the origin, which reads

$$\overline{\chi}_i(\zeta) = \chi_m + \zeta^{2i+2} + O(\zeta^{2i+4})\ ,\ i = 0, 1, 2, \ldots, \tag{9}$$

where we have normalized the coefficient of ζ^{2i+2} to one. This is consistent, since any solution of (1) is only determined up to a scale factor. The order of the minimum determines which solution is selected; each choice of i corresponds to one member in an infinite sequence of similarity solutions. Inserting (9) into (5), we obtain

$$\chi_m = \frac{1}{12(\overline{\delta} - 1)}, \quad \overline{T}_0 = \frac{1}{24}\frac{2\overline{\delta} - 1}{(\overline{\delta} - 1)^2}, \tag{10}$$

where $\overline{\delta} = (i + 1)\delta$. Note that the constants satisfy

$$\overline{T}_0/\chi_m^2 = 6(2\overline{\delta} - 1). \tag{11}$$

Normalizing $\overline{\chi}$ by its minimum,

$$\overline{\chi}(\zeta) = \chi_m f(\zeta), \tag{12}$$

the solution of (5) satisfying (9) and (7) is

$$\left(f + 2\overline{\delta} - 1\right)^{(2\overline{\delta}-1)/2} (f - 1)^{1/2} = \zeta^{i+1}. \tag{13}$$

To compute the integrals appearing in the constraint (8), we transform the integration variable from ζ to f, using

$$\int_{-\infty}^{\infty} \frac{d\zeta}{f^i} = \int_1^{\infty} \frac{df}{f_\zeta f^i},$$

where

$$f_\zeta^{-1} = \frac{\delta\zeta f}{(f + 2\overline{\delta} - 1)(f - 1)} = \delta f\, (f - 1)^{\frac{1}{2(i+1)} - 1} \left(f + 2\overline{\delta} - 1\right)^{\frac{2\overline{\delta}-1}{2(i+1)} - 1}. \tag{14}$$

Using (10), (8) can now be cast in the form

$$K_i(\delta) \equiv \int_1^{\infty} \left(\frac{2(1 - \overline{\delta})}{f^2} + \frac{2\overline{\delta} - 1}{f^3}\right)$$
$$\times \left(f + 2\overline{\delta} - 1\right)^{\frac{2\overline{\delta}-1}{2(i+1)} - 1} (f - 1)^{\frac{1}{2(i+1)} - 1} df = 0. \tag{15}$$

For each member $i = 0, 1, \ldots$ of the hierarchy of similarity solutions, (15) determines the corresponding scaling exponent δ_i.

The integrals appearing in (15) can be calculated analytically.[12] We introduce the notation

$$I(\alpha_1, \alpha_2, \alpha_3) \equiv \int_1^{\infty} x^{-\alpha_1}(x - 1)^{\alpha_2 - 1}(x + 2\overline{\delta} - 2)^{\alpha_3}$$
$$= B(\alpha_1 - \alpha_2 - \alpha_3, \alpha_2) F(-\alpha_3, \alpha_1 - \alpha_2 - \alpha_3; \alpha_1 - \alpha_3; -2\overline{\delta} - 1), \tag{16}$$

TABLE I. A list of exponents δ_i, found from $K_i(\delta) = 0$ using MAPLE, with K_i given by (17). The number $2i + 2$ gives the smallest non-vanishing power in a series expansion of the corresponding similarity solution around the origin. Only the solution with $i = 0$ is stable. The rescaled minimum radius is found from (10).

i	$\delta_i - 2$	χ_m
0	1.748717×10^{-1}	0.0709
1	4.538527×10^{-2}	0.0797
2	1.944257×10^{-2}	0.0817
3	1.053085×10^{-2}	0.0825
4	6.530260×10^{-3}	0.0828
5	4.419666×10^{-3}	0.0832

where B is the beta function and F the hypergeometric function. Thus, the function $K_i(\delta)$ may be written as

$$K_i(\delta_i) = 2(1-\bar{\delta})I\left(2, \frac{1}{2(i+1)}, \frac{2\bar{\delta}-1}{2(i+1)} - 1\right) + (2\bar{\delta}-1)I\left(3, \frac{1}{2(i+1)}, \frac{2\bar{\delta}-1}{2(i+1)} - 1\right). \tag{17}$$

Values for the exponents δ_i and the corresponding minimum radius χ_m are quoted in Table I.

III. STABILITY ANALYSIS

A. General

Following Giga and Kohn,[8] we perform a stability analysis of each of the similarity solutions found above by transforming to the new time variable $\tau = -\ln t'$ by virtue of

$$H = t' r_0 \chi(\zeta, \tau), \qquad T = t' T_0(\tau)\delta r_0. \tag{18}$$

Using the equation of motion (1), one finds the following dynamical system in similarity variables:

$$\chi_\tau = \frac{1}{6} + \chi - \delta\zeta\chi' - \frac{T_0(\tau)}{6\chi}. \tag{19}$$

The new formulation (19) is equivalent to the full thread dynamics (1). Its advantage is that any of the similarity solutions $\overline{\chi}$ are *fixed points* of (19), making a stability analysis much easier.

We introduce a perturbation around the fixed point solution by putting

$$\chi(\zeta, \tau) = \overline{\chi}(\zeta) + \epsilon\chi_m e^{\nu\tau} P(\zeta), \quad T_0 = \overline{T}_0 + \epsilon\overline{T}_0 e^{\nu\tau}\tilde{T}. \tag{20}$$

Linearizing (19) in ϵ, we obtain the eigenvalue equation

$$(\nu - 1)P = -\delta\zeta P_\zeta + \frac{\overline{T}_0 P}{6\chi_m^2 f^2} - \frac{\overline{T}_0\tilde{T}}{6\chi_m^2 f}. \tag{21}$$

Inserting (19) into (2), one obtains in the limit $t' \to 0$

$$T_0 = \left(\int_{-\infty}^{\infty}\frac{d\zeta}{\chi^3}\right)^{-1}\int_{-\infty}^{\infty}\frac{d\zeta}{\chi^4}. \tag{22}$$

Using the perturbation ansatz (20) in (22), one recovers (8) to leading order. The coefficient $\tilde{T}$, which represents the tension generated by the perturbation, is found comparing terms linear in ϵ:

$$\tilde{T} = -3\left(\int_{-\infty}^{\infty}\frac{d\zeta}{f^3}\right)^{-1}\int_{-\infty}^{\infty}\frac{P d\zeta}{f^4} + 4\left(\int_{-\infty}^{\infty}\frac{d\zeta}{f^4}\right)^{-1}\int_{-\infty}^{\infty}\frac{P d\zeta}{f^5}. \tag{23}$$

The eigenvalue equation (21) has to be solved subject to a growth condition on P at infinity, which ensures matching to an outer solution, similar to the condition (7) on $\overline{\chi}$.[9,11] Namely, the analogue of (6) now reads

$$H_t = r_0\left[-\overline{\chi} + \delta\zeta\overline{\chi}'\right] + \epsilon\chi_m e^{\nu\tau}\left[-P + \delta\zeta P' + \nu P\right]. \tag{24}$$

The first square bracket already vanishes for $\zeta \to \pm\infty$ owing to (7). For the second bracket to vanish as well, P has to satisfy the condition

$$P \propto \zeta^{(1-\nu)/\delta}. \tag{25}$$

Potential singularities of P occur at $\zeta = 0$. We thus have to impose the regularity condition

$$P \propto \zeta^{j}, \tag{26}$$

where $j = 0, 1, \ldots$. Note that since all similarity solutions are even, the eigenfunctions are either even or odd, so both even and odd powers appear in (26). To summarize, we have to find solutions $P(\zeta)$ of (21) with conditions (25) and (26), subject to the constraint (23). We will find that this is possible only for a discrete set of eigenvalues ν. For each similarity solution i, there will be an infinite sequence $\nu_j^{(i)}$.

For a similarity solution to be stable, all eigenvalues have to be negative. However, there always exist two positive eigenvalues which are not related to any instability but rather reflect the arbitrariness in the choice of t_0 and s_0.[13,14] Namely, any perturbation to the solution will also result in a small shift in the blowup time t_0 and blowup location s_0. If they are not adjusted accordingly, the variables t' and s' will not go to zero at the singularity, which appears as if one were driven away from the singularity. To derive the eigenvalues and eigenfunctions, consider that (4) is an equally good similarity solution if s' is shifted by an amount ϵ, giving

$$H^{(\epsilon)}(s', t') = r_0 t' \overline{\chi}\left(\frac{s' + \epsilon}{t'^{\delta}}\right) \equiv t' \chi^{(\epsilon)}(\zeta, \tau). \tag{27}$$

Expanding in ϵ, we obtain

$$\chi^{(\epsilon)}(\zeta, \tau) = \overline{\chi}(\zeta) + \epsilon t'^{-\delta} \overline{\chi}_{\chi} + O(\epsilon^2) \equiv \overline{\chi} + \epsilon e^{\delta\tau} \overline{\chi}_{\chi} + O(\epsilon^2). \tag{28}$$

Comparing to (20), one reads off that $\nu = \delta$ and $P = f_\zeta$. The tension does not change, and thus $\tilde{T} = 0$. Similarly, defining a time-translated solution by

$$H^{(\epsilon)}(s', t') = r_0(t' + \epsilon)\overline{\chi}\left(\frac{s'}{(t' + \epsilon)^{\delta}}\right) \equiv t' \chi^{(\epsilon)}(\zeta, \tau), \quad T^{(\epsilon)} = (t' + \epsilon)\overline{T}_0 r_0^2 \equiv t' T_0^{(\epsilon)}(\tau) r_0^2, \tag{29}$$

the linearized version becomes

$$\chi^{(\epsilon)}(\zeta, \tau) = \overline{\chi}(\zeta) + \epsilon e^{\tau}\left(\overline{\chi}_\zeta - \delta\zeta\overline{\chi}_\zeta\right), \quad T_0^{(\epsilon)}(\tau) = \overline{T}_0 + \epsilon\overline{T}_0 e^{\tau}. \tag{30}$$

Once more comparing to (20), the eigenvalue is $\nu = 1$ with eigenfunction $P = f - \delta\zeta f_\zeta$ and $\tilde{T} = 1$. Finally, we recall that all solutions are determined only up to an arbitrary scale factor a multiplying ζ, since the original equation (1) is invariant under a change of scale in s. The corresponding eigenvalue must be $\nu = 0$, and the eigenfunction

$$\left.\frac{\partial f(a\zeta)}{\partial a}\right|_{a=1} = \zeta f_\zeta = \frac{(f + 2\overline{\delta} - 1)(f - 1)}{\delta f}. \tag{31}$$

The calculations below will be performed writing P as function of f instead of ζ; transformation between the two are achieved using (13) and (14). The three eigenvalues and eigenfunctions related to invariances are listed in Table II, using f as the independent variable. Converting the eigenvalue equation (21) to f as the independent variable, we find

$$(\nu - 1)P = -\frac{(f + 2\overline{\delta} - 1)(f - 1)}{f}\frac{dP}{df} + \frac{2\overline{\delta} - 1}{f^2}P - \frac{(2\overline{\delta} - 1)\tilde{T}}{f}. \tag{32}$$

It is a simple matter to verify that the eigenfunctions listed in Table II indeed solve (32) with the correct eigenvalues.

In the homogeneous case $\tilde{T} = 0$, the solution of (32) is

$$\overline{P} = \frac{1}{f}\left(f + 2\overline{\delta} - 1\right)^{1 - \frac{\nu(2\overline{\delta}-1)}{2\overline{\delta}}}(f - 1)^{1 - \frac{\nu}{2\overline{\delta}}}, \tag{33}$$

TABLE II. Eigenfunctions and eigenvalues derived from symmetries of the equations.

Eigenvalue	Eigenfunction	Tension	Parity	Invariance
$\nu = \delta_i$	$P = (f + 2\overline{\delta_i} - 1)^{1-\delta_i+1/(2(i+1))}(f-1)^{1-1/(2(i+1))}/f$	$\tilde{T} = 0$	Odd	Time
$\nu = 1$	$P = -(2((i+1)\delta_i - 1)f + 1 - 2(i+1)\delta_i)/f$	$\tilde{T} = 1$	Even	Space
$\nu = 0$	$P = (f + 2(i+1)\delta_i - 1)(f-1)/f$	$\tilde{T} = 0$	Even	Axial scale

while the general solution is

$$P = \overline{P}\left\{C + \tilde{T}(1 - 2\overline{\delta})\int_1^f A(f)df\right\}, \tag{34}$$

where we define

$$A(f) = f\left(f + 2\overline{\delta} - 1\right)^{\frac{\nu(2\overline{\delta}-1)-4\overline{\delta}}{2\overline{\delta}}}(f-1)^{-2+\nu/(2\overline{\delta})} \equiv (f-1)^{-2+\nu/(2\overline{\delta})}A_1 \tag{35}$$

for later convenience.

IV. THE EIGENVALUE SPECTRUM

We will consider two cases separately: solutions with $\tilde{T} = 0$ and $\tilde{T} \neq 0$. In the former case, it is sufficient to consider the homogeneous solution (33). We will show that the only solution with $\tilde{T} \neq 0$ is in fact that with eigenvalue $\nu = 1$ listed in Table II. Thus, all of the work in this case consists in showing that there are no *other* solutions.

A. No tension

We begin by discussing the case $\tilde{T} = 0$. The asymptotics of $\overline{P}$ are

$$\overline{P}\Big|_{f\to 1} \propto (f-1)^{1-\frac{\nu}{2\overline{\delta}}}, \quad \overline{P}\Big|_{f\to\infty} \propto f^{1-\nu}. \tag{36}$$

From (13), the behavior of f_i at the origin and for large ζ is given by

$$f - 1|_{\zeta\to 0} \propto \zeta^{2(i+1)}, \quad f|_{\zeta\to\infty} \propto \zeta^{1/\delta}. \tag{37}$$

Thus, the growth condition (25) is satisfied automatically.

The behavior of P in the origin is

$$P \propto \zeta^{2(i+1)\left(1-\frac{\nu}{2\overline{\delta}}\right)},$$

so according to (26) we find that the eigenvalues must satisfy

$$\nu_j^{(i)} = \delta_i(2i + 2 - j), \quad j = 0, 1, 2, \ldots. \tag{38}$$

Two eigenvalues from this series, $\nu^{(i)} = \delta_i$ and 0, correspond to known values listed in Table II, with $\tilde{T} = 0$.

Now we must look at the constraint (23), which should give $\tilde{T} = 0$. Clearly, this is satisfied identically if j is odd, since the integral over P vanishes. As for even values of j, $\nu = 0$ for $j = 2i + 2$, which has vanishing tension according to Table II. All other even values lead to finite tension and are therefore excluded, as we will see now. With the transformation $d\zeta = df/f_\zeta$, (23) is converted to

$$\tilde{T} = -3\left(\int_1^\infty \frac{df}{f_\zeta f^3}\right)^{-1}\int_1^\infty \frac{Pdf}{f_\zeta f^4} + 4\left(\int_1^\infty \frac{df}{f_\zeta f^4}\right)^{-1}\int_1^\infty \frac{Pdf}{f_\zeta f^5}. \tag{39}$$

Using (14) as well as (36), one finds that for large f

$$\frac{P}{f_\zeta f^4} \propto f^{-\nu-4+\delta}.$$

TABLE III. The first few eigenfunctions and eigenvalues for the ground state ($i = 0$) and the first excited state ($i = 1$). Solutions listed in Table II are not included.

Eigenvalue	Eigenfunction	Tension	Parity	Solution	Number
$\nu_3^{(0)} = -\delta_0$	$P = (f + 2\delta_0 - 1)^{1/2+\delta_0}(f-1)^{3/2}/f$	$\tilde{T} = 0$	odd	i = 0	j = 3
$\nu_0^{(1)} = 3\delta_1$	$P = (f + 4\delta_1 - 1)^{7/4-3\delta_1}(f-1)^{1/4}/f$	$\tilde{T} = 0$	odd	i = 1	j = 1
$\nu_5^{(1)} = -\delta_1$	$P = (f + 2\delta_0 - 1)^{1/2+\delta_1}(f-1)^{5/4}/f$	$\tilde{T} = 0$	odd	i = 1	j = 5

In other words, the first P-integral on the right of (39) diverges if $-\nu - 4 + \delta \geq -1$. This means to have a finite tension $\tilde{T}$, the eigenvalue must satisfy $\nu > \delta - 3$. Since δ satisfies $2 < \delta < 3$, in view of (38) this means that

$$0 \leq j \leq 2i + 3. \tag{40}$$

Thus, for each i, there remain a finite number of even values of j for which to evaluate $\tilde{T}$; using (16), we find

$$\tilde{T} = -3\frac{I\left(4, 1 - \frac{\nu}{2\delta} + \frac{1}{2(i+1)}, \frac{2\bar{\delta}-1}{2(i+1)} - \nu + \frac{\nu}{2\delta}\right)}{I\left(2, \frac{1}{2(i+1)}, \frac{2\bar{\delta}-1}{2(i+1)} - 1\right)} + 4\frac{I\left(5, 1 - \frac{\nu}{2\delta} + \frac{1}{2(i+1)}, \frac{2\bar{\delta}-1}{2(i+1)} - \nu + \frac{\nu}{2\delta}\right)}{I\left(3, \frac{1}{2(i+1)}, \frac{2\bar{\delta}-1}{2(i+1)} - 1\right)}. \tag{41}$$

For the first few similarity solutions, we found that $\tilde{T} \neq 0$ for each even j satisfying (40), contradicting the assumption of vanishing tension. In conclusion, for $\tilde{T} = 0$ the first two eigenvalue series are

$$\nu^{(0)} = \delta_0, 0, -\delta_0, -3\delta_0, \ldots \quad \text{(ground state)},$$
$$\nu^{(1)} = 3\delta_1, \delta_1, 0, -\delta_1, \ldots . \quad \text{(first excited state)}.$$

In particular, this means that the similarity solution with $i = 1$ (which according to (9) has a quartic minimum), is *unstable* on account of the positive eigenvalue $\nu_1^{(1)} = 3\delta_1$. The other positive eigenvalues of both series appear in Table II and do not correspond to instability. Higher order similarity solutions, with higher order minima, of course have even more unstable eigenvalues. This disproves the conjecture made in Ref. 5 that all similarity solutions are linearly stable. It remains to be shown that the ground state solution is linearly stable by calculating all eigenvalues with $\tilde{T}$ finite.

Eigenfunctions and eigenvalues for $i = 0$, 1 (the ground state and the first excited state) are summarized in Table III, *excluding* those modes which are listed in Table II, and which do not cause instability. In Fig. 1, the same eigenfunctions are plotted as function of the variable ζ. The eigenfunction in the middle is responsible for the linear instability of the first excited state. It has a finite width in the similarity variable, and thus has the same axial scale as the underlying similarity solution. Hence, it is not captured by the stability analysis in Ref. 5.

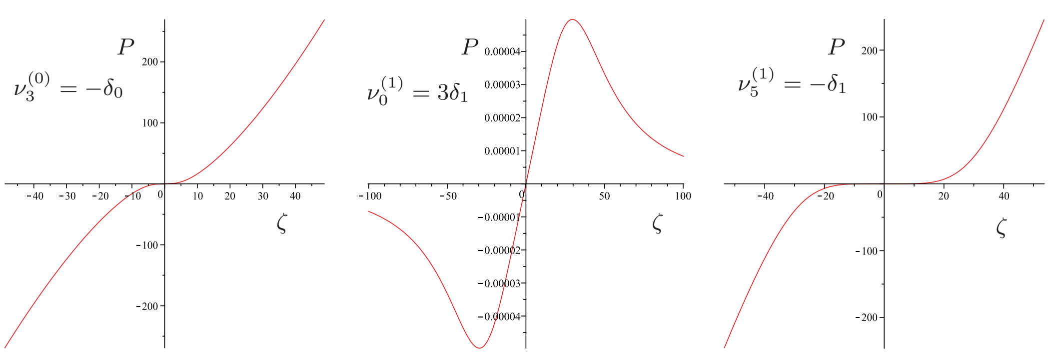

FIG. 1. The eigenfunctions corresponding to Table III.

B. Finite tension

Now we consider the case of $\tilde{T}$ finite. As is seen from Table II, there exists one such eigenfunction, which has the eigenvalue $\nu = 1$, and which comes from invariance under shifts in t_0. The purpose of this section is to show that for each similarity solution, this is the only eigenfunction with finite tension.

We begin by observing that an eigensolution P with finite tension must be proportional to $\tilde{T}$, otherwise it would not be possible to satisfy the constraint (23). In other words, P must be of the form (34) with $C = 0$. To find the expansion of this solution near $f = 1$, we go back to the differential equation (32), only taking into account terms proportional to $\tilde{T}$. This results in the expansion

$$P = \sum_{j=0}^{\infty} a_j (f-1)^j, \tag{43}$$

with coefficients

$$a_0 = \frac{2\overline{\delta} - 1}{2\overline{\delta} - \nu}\tilde{T}, \quad a_1 = \frac{(2\overline{\delta} - 1)(2 - 2\overline{\delta} - \nu)}{(2\overline{\delta} - \nu)\nu}\tilde{T}, \ldots. \tag{44}$$

In other words, the solutions $P(\zeta)$ we are seeking are all regular at the origin $\zeta = 0$, and must be even, since $f(\zeta)$ is even. Thus, as before it follows that $\nu > \delta - 3$, otherwise the first integral over P on the right of (39) would be infinite.

Next, we consider the asymptotic condition (25), which is satisfied identically by the homogeneous solution $\overline{P}$. This means that the integral over A in (34) must be convergent at the upper limit, otherwise P would grow too quickly at infinity. The asymptotics of A at infinity are

$$A|_{f\to\infty} \propto f^{\nu-3}, \tag{45}$$

and so we must have $\nu < 2$ for the integral to convergent. In other words, the eigenvalue must satisfy

$$\delta - 3 < \nu < 2. \tag{46}$$

The behavior of A near $f = 1$ is

$$A|_{f\to 1} \propto (f-1)^{-\frac{4\overline{\delta}-\nu}{2\overline{\delta}}}, \tag{47}$$

so the integral in (34) is divergent at its lower limit if $\nu \leq 2\overline{\delta}$. To make the integral convergent over the whole range of eigenvalues given by (46), we integrate by parts twice, turning (34) into

$$\tilde{P} = \overline{P}(1 - 2\overline{\delta})\left\{\frac{(f-1)^{-1+\nu/(2\overline{\delta})}}{-1+\nu/(2\overline{\delta})}A_1 - \frac{2\overline{\delta}(f-1)^{\nu/(2\overline{\delta})}}{(-1+\nu/(2\overline{\delta}))\nu}A_1' + \int_1^f \frac{2\overline{\delta}(f-1)^{\nu/(2\overline{\delta})}}{(-1+\nu/(2\overline{\delta}))\nu}A_1'' df\right\}, \tag{48}$$

where $\tilde{P}$ is the eigenfuncton normalized by the tension $\tilde{T}$.

Now we must check that (48) satisfies the constraint (39), which means that

$$\mathcal{F}(\nu) \equiv -3\left(\int_1^\infty \frac{df}{f_\zeta f^3}\right)^{-1}\int_1^\infty \frac{\tilde{P}df}{f_\zeta f^4} + 4\left(\int_1^\infty \frac{df}{f_\zeta f^4}\right)^{-1}\int_1^\infty \frac{\tilde{P}df}{f_\zeta f^5} = 1. \tag{49}$$

One solution of (49) is $\nu = 1$, and the corresponding P is found in Table II. Namely, for $\nu = 1$, (48) turns into

$$P = -(2(\overline{\delta} - 1)f + 1 - 2\overline{\delta})/f,$$

and all integrals can be performed, using (16). Thus one confirms that (49) is indeed satisfied for $\nu = 1$.

The remaining question is whether (49) has *other* solutions in the interval (46). We were not able to calculate $\mathcal{F}(\nu)$ analytically, so the relevant integrals were done numerically instead. Some details of this calculation are reported in the Appendix, and the resulting graph of $\mathcal{F}(\nu)$ is plotted in Fig. 2. We focus on the case $i = 0$, since we want to make sure there are no positive eigenvalues we might have missed; results for $i = 1$ are similar. Clearly, $\mathcal{F}$ is a monotonic function over the relevant

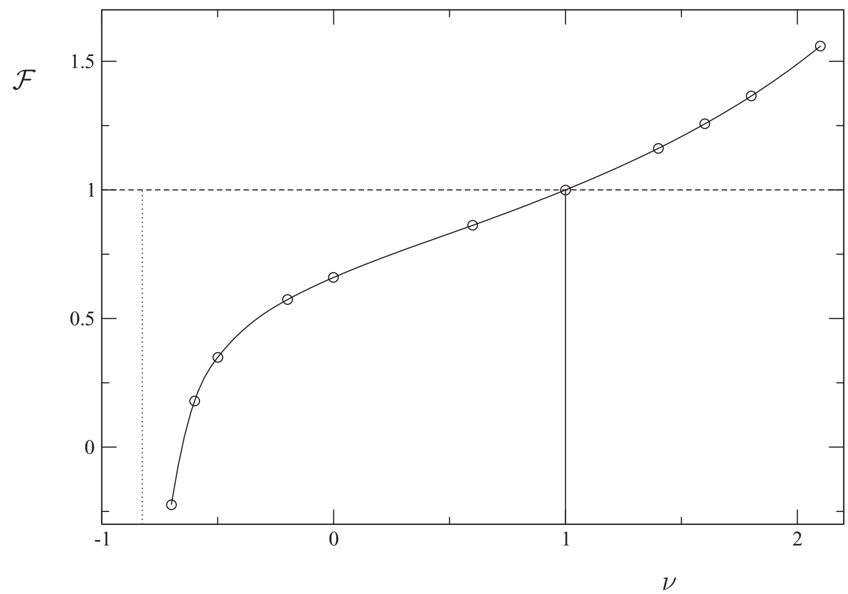

FIG. 2. The right hand side of (23) as function of the eigenvalue ν, for the similarity solution $i = 0$; $\mathcal{F}(1) = 1$. No solution with the right asymptotics exists for $\nu > 2$, and for $\nu \to \delta - 3$ (vertical dotted line) the expression diverges to $-\infty$.

interval, and diverges to $-\infty$ as $\nu \to \delta - 3$. Thus, (43) is a complete list of the eigenvalues of the ground state. The first two eigenvalues δ_0 and 0 do not cause instability, all other eigenvalues are negative. Thus, the ground state solution is linearly stable.

V. DISCUSSION

So far, our calculation was done entirely in Lagrangian coordinates. To transform back to physical (Eulerian) coordinates, we note that

$$z = \int_0^s H^{-2}(\overline{s}, t) d\overline{s}. \tag{50}$$

This implies that a typical axial scale is $\Delta z \propto \Delta t^{\delta - 2}$, and thus the similarity solution is

$$h(z, t) = r_0 t' \phi_{St}^{(i)} \left(\frac{z}{r_0 t'^{\delta_i - 2}} \right). \tag{51}$$

Converting (50) to similarity variables, one finds

$$\xi = \frac{1}{\chi_m^2} \int_0^{\zeta} \frac{d\overline{\zeta}}{f62(\overline{\zeta})} = \frac{\delta}{\chi_m^2} \int_1^f \frac{(\overline{f} + 2\overline{\delta}_i - 1)^{\frac{2\overline{\delta}_i - 1}{2(i+1)} - 1}}{\overline{f}(\overline{f} - 1)^{1 - \frac{1}{2(i+1)}}} d\overline{f}. \tag{52}$$

Using (52), eigensolutions $P(f)$ can be converted to an Eulerian description $P(\xi)$. Written as an equation for the profile f as function of ξ, (52) describes the shape of the similarity solutions in real space.

As can be seen from Table I, the Eulerian exponent $\delta_0 - 2$ corresponding to the stable ground state solution is quite small. This means that the fluid thread becomes very long and thin near the pinch point, since axial scales are much *larger* than radial scales. For the (unstable) higher order solutions, $\delta_1 - 2$ is still smaller. In addition, the profile has a quartic minimum

$$\phi_{St}^{(1)} - \chi_m^{(i)} \propto \xi^4,$$

making it appear even more flat.

In Ref. 11, we advanced a qualitative argument as to why only the ground state solution is stable, while all higher order solutions are unstable. Namely, the ground state solution corresponds to a generic minimum, while for the quadratic term to vanish (quartic solutions and higher), a particular initial condition is necessary. Thus, if the quartic solution is realized, a small perturbation will make the quadratic coefficient nonzero, and drive the evolution away from the quartic profile. This explains why all similarity solutions except for the ground state solution are unstable. However, this argument is by no means rigorous, and an explicit calculation was necessary to prove the point. For example, the above "generic" argument would suggest that with each step in the hierarchy of similarity solutions, two more modes, one even and one odd, become unstable. However, the explicit solution shows that the even solutions are excluded, since they are inconsistent with the constraint on the tension in the thread.

There are a number of examples of singularities for which an infinite sequence of solutions exist, but only one of them (the "ground state") is stable, all others are linearly unstable. One such example studied numerically[9] is the surface-tension-driven breakup of a solid cylinder with surface diffusion. Another is the development of a shock in Burgers' equation.[11] In the case of fluid breakup with inertia, an infinite set of similarity solutions has been found.[5] Higher order solutions are always found to destabilize, but whether this is the result of a linear instability or a nonlinear instability with a very low threshold remains to be settled by a proper linear stability calculation. It is tempting to believe that the fact that precisely one solution out of an infinite sequence is stable is not coincidental, but is the result of the ground state solution being generic in an appropriate space of solutions. It remains to be seen if the non-genericity in other solution hierarchies can be discovered, and turned into a tool to understand the stability properties of the entire hierarchy.

APPENDIX: CALCULATION OF $\mathcal{F}(\nu)$

Here, we make sure that (49) only has a single solution in the interval $\delta - 3 < \nu < 2$, as seen in Fig. 2. We confine ourselves to the ground state solution $i = 0$. For $\nu = \delta - 3 \approx -0.825$, the integral

$$\int_1^\infty \frac{\tilde{P} df}{f_\zeta f^4}$$

diverges logarithmically, which means that $\mathcal{F}$ goes to $-\infty$. At the upper limit $\nu = 2$, $\mathcal{F}$ remains regular, but eigenvalues $\nu \geq 2$ are disallowed, since $\tilde{P}$ grows too quickly at infinity.

First, $\tilde{P}(f)$ is calculated using (48), performing the integral numerically. Next, the definite integrals over $\tilde{P}/(f_\zeta f^4)$ and $\tilde{P}/(f_\zeta f^5)$ have to be performed. Finally,

$$\int_1^\infty \frac{df}{f_\zeta f^n} = \delta I\left(n-1, \frac{1}{2(i+1)}, \frac{2\overline{\delta}-1}{2(i+1)} - 1\right),$$

using (16). It is a good check on the numerics to calculate $\mathcal{F}(1)$, for which we find 1 to an error of 10^{-6}.

As is apparent from (48), the expressions become singular for $\nu \to 0$, so we have to perform this limit more carefully. If we write

$$\frac{2\overline{\delta}}{\nu/(2\delta) - 1}\frac{dA_1}{df} = B_0(f) + \nu B_1(f) + O(\nu^2), \quad \overline{P} = \overline{P}_0 + \nu \overline{P}_1 + O(\nu^2),$$

then the expansion of $\tilde{P}$ is

$$\tilde{P} = \frac{2\overline{\delta}-1}{\nu} B_0(1)\frac{f-1}{f}\left(f + 2\overline{\delta} - 1\right) + P_{fin} + O(\nu),$$

where the finite part is

$$P_{fin} = \overline{P}_0(1-2\overline{\delta})\left\{\frac{f}{(f-1)\left(f+2\overline{\delta}-1\right)^2} - B_1(1) - \frac{\ln(f-1)B_0}{2\overline{\delta}} + \frac{1}{2\overline{\delta}}\int_1^f \ln(f-1)B_0'(f)df\right\} + (2\overline{\delta}-1)B_0(1)\overline{P}_1.$$

Comparing to Table II, one observes that the part which is singular in ν is the eigenfunction with eigenvalue zero, which corresponds to $\tilde{T} = 0$. Thus, the contribution of the singular part vanishes identically, and we have to include the next order. Evaluating the integral, the finite part becomes

$$P_{fin} = -\frac{2\overline{\delta} - 1}{2f\overline{\delta}^2}\Big\{a_0 + \ln(f + 2\overline{\delta} - 1)(2\overline{\delta}^2 - 3\overline{\delta} + 1 + f(4\overline{\delta} - 2 - 2\overline{\delta}^2) + f^2(1 - \overline{\delta})) + a_1 f + a_2 f^2\Big\}, \tag{A1}$$

where

$$a_0 = 1 - \ln\overline{\delta} - \ln 2 + \overline{\delta}(3\ln\overline{\delta} + 3\ln 2 - 2) - 2\overline{\delta}^2(\ln\overline{\delta} - \ln 2),$$
$$a_1 = 2\ln\overline{\delta} + 2\ln 2 - 1 - 4\overline{\delta}\ln 2 + 2\overline{\delta}^2\ln\overline{\delta} - 4\overline{\delta}\ln\overline{\delta} + 2\overline{\delta}^2\ln 2 + \overline{\delta},$$
$$a_2 = -\ln 2 - \ln\overline{\delta} + \overline{\delta}\ln\overline{\delta} + \overline{\delta}\ln 2.$$

Note that P_{fin} is regular at the origin and has the expected asymptotics (25), consistent with $\nu = 0$. However, P_{fin} does not solve the eigenvalue equation (21), since it results from a singular limit. Inserting P_{fin} into the right hand side of (39) and evaluating the integrals numerically gives 0.6597, consistent with the numerics for finite ν, see Fig. 2.

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