Universal Pinching of 3D Axisymmetric Free-Surface Flow

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We consider the viscous motion of an axisymmetric column of fluid with a free surface. The Navier-Stokes equation forms a singularity as the height of the fluid neck goes to zero. Close to pinchoff, the solutions have a scaling form characterized by a set of universal exponents. The shape of the neck and its velocity field is described by scaling functions, which we predict without adjustable parameters.

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Much of the current interest in singularities of partial differential equations [1–4] has been generated not only by their physical significance, but also by the expectation that their behavior is governed by universal scaling laws, largely independent of boundary or initial conditions. The physical reasoning is that singularities involve the production of infinitely small scales, so that the equations locally evolve on scales widely separated from the ones set by the boundary conditions. Thus one is led to universality and, in the absence of any length scale, self-similarity near the singularity.

In this Letter we establish scaling solutions for the pinching of an axisymmetric fluid neck. Typical physical examples are the breakup of jets [5] or the dripping of a faucet [6]. Linear stability analysis of those problems, which describes the onset of drop formation, already has its own and long-standing history [7,8] . However, very little is known about the last stages of drop formation, when the neck behind a drop becomes arbitrarily small, and fluid is expelled from it with increasingly high speed. Quite strikingly, the theory presented here not only predicts universal exponents, but also universal amplitudes for the scaling functions. This means the neck thickness or the fluid velocity inside the neck can be predicted at a given time distance from the singularity, independent of the experiment considered. This is because the relevant length and time scales are set completely by *internal* properties of the equations, namely, by a balance between surface tension forces trying to reduce the neck radius, and viscous forces. To pick an arbitrary example, consider the pinching of glycerol at 20°C. Independent of the type of experiment, 0.01 sec away from the singularity the minimum neck radius will be $h_{\min} = 13.1\ \mu$m, and the maximum fluid velocity $v_{\max} = 106$ cm/sec.

This implies a higher degree of universality as previously known in scaling theories. In critical phenomena [9], only amplitude *ratios* are universal. In previous studies of two-dimensional pinching [2], the singularity has to be provoked by the boundary conditions, and therefore amplitudes depend on the applied pressure, or other properties of the boundary.

The Navier-Stokes equation for an axisymmetric column of incompressible fluid with viscosity η, density ρ, and surface tension γ reads [10]

$$\rho(\partial_t v_r + v_r\partial_r v_r + v_z\partial_z v_r) \\ = -\partial_r p + \eta(\partial_r^2 v_r + \partial_z^2 v_r + \partial_r v_r/r - v_r/r^2), \quad (1)$$

$$\rho(\partial_t v_z + v_r\partial_r v_z + v_z\partial_z v_z) \\ = -\partial_z p + \eta(\partial_r^2 v_z + \partial_z^2 v_z + \partial_r v_z/r) - \rho g, \quad (2)$$

and

$$\partial_r v_r + \partial_z v_z + v_r/r = 0, \quad (3)$$

where v_z is the velocity along the axis, v_r the velocity in the radial direction, and p the pressure. The boundary conditions are

$$\mathbf{n}\,\sigma\,\mathbf{n} = -\gamma(1/R_1 + 1/R_2) \quad (4)$$

and

$$\mathbf{n}\,\sigma\,\mathbf{t} = 0. \quad (5)$$

Here σ is the stress tensor, $\mathbf{n}$ the outward normal, and R_1 and R_2 are the principal radii of curvature. The equation of motion for the height of the fluid neck $h = h(z,t)$ is

$$\partial_t h + v_z h' = v_r|_{r=h}, \quad (6)$$

where a prime refers to differentiation with respect to z.

A crucial simplification comes in by keeping only the lowest order radial dependence in (1)–(6). To this end the velocity and pressure fields are expanded into a power series in r: $v_z(z,r,t) = v_0(z,t) + v_2(z,t)r^2 + \cdots$, $p(z,r,t) = p_0(z,t) + p_2(z,t)r^2 + \cdots$, and, in accordance with (3), $v_r(z,r,t) = -v_0'(z,t)r/2 - v_2'(z,t)r^3/4 - \cdots$. We insert this ansatz into (1)–(6) and in each case only take the lowest order terms in r and h into account. Then (2) gives

$$\rho(\partial_t v_0 + v_0 v_0') = -p_0' + \eta(4v_2 + v_0'') - \rho g, \quad (7)$$

and (1) is satisfied identically. The unknown functions p_0 and v_2 are eliminated from (7) by using the boundary conditions (4) and (5), respectively: $p_0 = \gamma(1/R_1 + 1/R_2) - \eta v_0'$ and $v_2 = v_0''/4 + (3/2)v_0'h'/h$. This leaves us with a coupled set of equations for the velocity $v(z,t) \equiv v_0(z,t)$ and the height $h(z,t)$:

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$$\rho(\partial_t v + vv') = -\gamma \left(\frac{1}{R_1} + \frac{1}{R_2}\right)' + 3\eta \frac{(h^2 v')'}{h^2} - \rho g, \quad (8)$$

$$\partial_t h + vh' = -v'h/2. \quad (9)$$

For the moment we keep the complete expression for the mean curvature,

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{h(1+h'^2)^{1/2}} - \frac{h''}{(1+h'^2)^{3/2}}. \quad (10)$$

We will see below that close to the singularity higher order terms involving $v_2(z,t)$, etc., become arbitrarily small, so that (8) and (9) are an *exact* representation of (1)–(6) asymptotically. But even away from the singularity (8) and (9) give an excellent description of experiments. This is demonstrated in [11] for both jets and the dripping faucet.

The natural length and time units of the singularity are $\ell_\nu = \eta^2/\gamma\rho$ and $t_\nu = \eta^3/\gamma^2\rho$, which are the scales where viscous and surface tension forces are balanced. Length and time distances from the singularity are measured by $z' = (z - z_0)/\ell_\nu$ and $t' = (t_0 - t)/t_\nu$, respectively. In the pinch region, meaning that $|z'| \ll 1$ and $t' \ll 1$, we expect $h(z,t)$ and $v(z,t)$ to have the scaling form

$$h(z,t) = \ell_\nu t'^{\alpha_1} \phi(z'/t'^{\beta}), \quad (11)$$
$$v(z,t) = (\ell_\nu/t_\nu) t'^{\alpha_2} \psi(z'/t'^{\beta}).$$

The correct exponents α_1, α_2, and β in (11) can be inferred from dimensional analysis [12] by demanding certain regularity properties of the functions h and v as one of the physical parameters η, γ, or ρ goes to zero. This corresponds to saying which terms in (8) remain relevant as $t' \to 0$. Surprisingly, the term to be dropped is $\gamma[h''/(1+h'^2)^{3/2}]'$, which contains the highest derivative in the problem. It comes from the radius of curvature in the direction of the axis. This means we demand $h \sim \gamma$, and v to be finite as $\gamma \to 0$, so that $-\gamma(1/R_1 + 1/R_2)' \to \gamma h'/h^2$ and all other terms in (8) remain finite in the limit. There is only one particular combination of η, γ, and ρ consistent with this requirement, giving $\alpha_1 = 1$, $\alpha_2 = -1/2$, and $\beta = 1/2$. All other choices turn out to be inconsistent, in that the term dropped is more singular than the ones retained as $t' \to 0$, or either h or v are *not* singular as $t' \to 0$.

Inserting (11) into (8) and (9) leads to a coupled set of ordinary differential equations for the scaling functions $\phi(\xi)$ and $\psi(\xi)$ in the similarity variable $\xi = z'/t'^{1/2}$:

$$\psi/2 + \xi\psi'/2 + \psi\psi' = \phi'/\phi^2 + 3\psi'' + 6\psi'\phi'/\phi, \quad (12)$$

$$\phi' = \phi \frac{1 - \psi'/2}{\psi + \xi/2}. \quad (13)$$

As $t' \to 0$, the highest order singularities in (8) are proportional to $t'^{-3/2}$. All lower order terms have been dropped in (12), since they do not affect the asymptotic behavior. Specifically, gravity only contributes a constant acceleration and therefore drops out of the problem.

The solutions of (12) and (13) are parametrized by a three-dimensional manifold of initial conditions. However, a set of consistency conditions selects precisely one pair of profiles ϕ, ψ which is physically realizable. The first condition comes from the fact that the profiles $h(z,t)$ and $v(z,t)$ are expected to move slowly away from the pinch point. This means the critical time dependence on t' should cancel as $z' \to \pm\infty$ [3], leading to two mutually consistent conditions,

$$\phi(\xi) \to \xi^{\alpha_1/\beta} = \xi^2, \quad (14)$$
$$\psi(\xi) \to \xi^{\alpha_2/\beta} = \xi^{-1},$$

as $\xi \to \pm\infty$. By inserting the expansion of ϕ and ψ around $\xi = \pm\infty$ into (12) and (13) one indeed finds that solutions to (12) and (13) with the correct asymptotics as $\xi \to \pm\infty$ are parametrized by only two free coefficients, respectively.

The remaining third condition is found from (13). Since $\psi(\xi)$ decays at infinity, it must be bounded. This means the denominator $\psi + \xi/2$ in (13) will become zero at some point ξ_0. Demanding ϕ to be analytic on the real axis amounts to saying that $\psi'(\xi_0) = 2$, which is the missing third condition. This leaves us with power series expansions of ϕ and ψ around $\xi = -\infty$, ξ_0, and $+\infty$, each of which is parametrized by two real numbers. Since the radii of convergence do not overlap, (12) and (13) have to be integrated numerically in between those regions. Using a shooting method, one finds the unique profiles represented in Fig. 1 by solid lines. In particular, there is no symmetric solution. By taking the total time derivative of $h(z - z_{\min}(t), t) = \ell_\nu t' \phi(\xi - \xi_{\min})$ one therefore finds that the minimum of $h(z,t)$ moves with the nonzero velocity $\dot{z}_{\min}(t) = (\ell_\nu/t_\nu)(-\xi_{\min}/2)t'^{-1/2}$. This also provides us with a physical interpretation of the special point ξ_0: It corresponds to the point where the fluid is at rest in the frame of reference of the moving surface.

To check if the pinch solutions are actually stable, we numerically integrated (8) and (9) as described in [11]. For all our runs, covering a wide range of viscosities and different initial conditions, we always found the solutions to scale like (11) as $t' \to 0$, the resulting scaling functions ϕ, ψ collapsing onto a single curve.

In Fig. 1 we show the convergence of numerical solutions towards the similarity solution as t' goes to zero, for a typical run in the jet geometry. The dashed line corresponds to $t' = 0.39$, t' being smaller by a factor of 3 for the chain-dashed, dot-dashed, and dotted lines, respectively. We expect the similarity solution to be valid for $|z'| \lesssim 1$ and $t' \lesssim 1$, while outside of this region higher order corrections become important. In the similarity variable $\xi = z'/t'^{1/2}$ this means the range of validity should *expand* like $t'^{-1/2}$. This is seen rather nicely in Fig. 1, as the ξ value at which the simulation starts to deviate

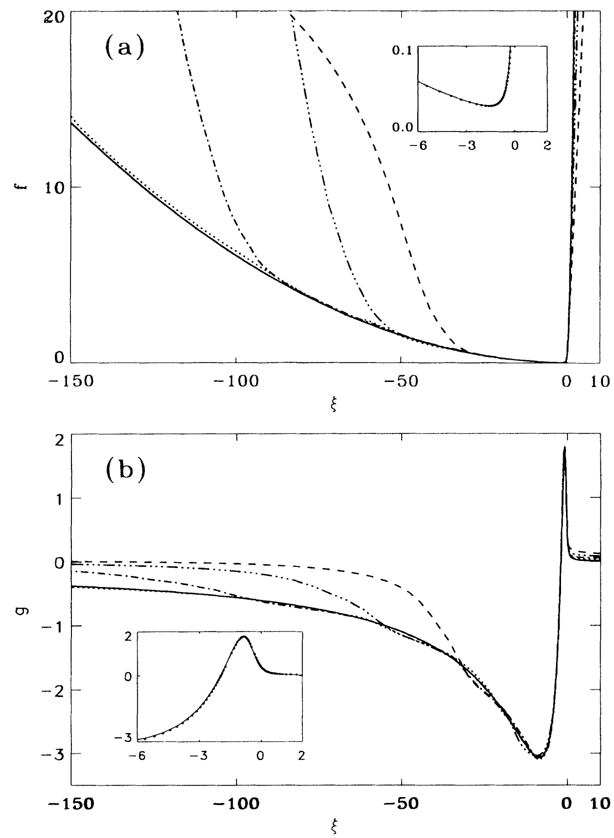

FIG. 1. The scaling functions of the height, ϕ, and of the velocity, ψ, close to pinchoff. The full curve is the prediction of the present theory; the dashed, chain-dashed, dot-dashed, and dotted lines represent the result of a simulation at $t' = 0.39, 0.13, 0.043$, and 0.014. The inset contains a blowup of the central region with only the latest time, $t' = 0.014$. Note that there is no adjustable parameter in this comparison.

from theory is larger by about a factor of $\sqrt{3} \approx 1.7$ as t' decreases by a factor of 3. The inset demonstrates the excellent agreement in the central pinch region for $t' = 0.014$ (dotted line). Note that there is no adjustable parameter in this comparison.

We mention that preliminary calculations [13] at very high viscosities and in the presence of gravity seem to show similarity form only over a finite range of ξ values. Under those conditions, the fluid forms thin threads, which appear to favor secondary instabilities leading to breakdown of the similarity solution for $|\xi|$ greater than some critical value.

It follows from (11) that the width of the pinch region scales like $t'^{1/2}$, whereas h goes to zero like t'. Therefore, if the profile $h(z,t)$ is normalized by the minimum height, the pinch region forms a long and thin neck as $t' \to 0$. As one can check explicitly by carrying out the expansion in the radial coordinate to higher order, this is the reason corrections to (8) and (9) vanish as the neck is pinching off and the solution applies to the full axisymmetric Navier-Stokes equations (1)–(6).

To demonstrate the predictive power of the theory, we calculated the minimum height $h_{\min}$, maximum velocity $v_{\max}$, and velocity of the minimum $\dot{z}_{\min}$ from ϕ and ψ. The result is $h_{\min} = 0.0304(\gamma/\eta)(t_0 - t)$, $v_{\max} = 3.07(\eta/\rho)^{1/2}(t_0 - t)^{-1/2}$, and $\dot{z}_{\min} = 0.80(\eta/\rho)^{1/2}(t_0 - t)^{-1/2}$. Applied to the pinching of glycerol at 20°C ($\ell_\nu = 2.79$ cm, $t_\nu = 0.65$ sec), we find the numbers given in the beginning of this Letter. Experiments to verify the predictions of the theory are under way [14].

Many aspects of this problem remain to be investigated. The most interesting seems to be the question of unique continuation of the Navier-Stokes equation to times *after* the singularity. The existence of a universal similarity solution for $t' < 0$ would imply that the breaking of a fluid neck is independent of the microscopic structure of the fluid, in other words, that breaking is a hydrodynamic phenomenon.

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