MTI Exercises 5: Solutions

1. (i) The inequality $0 \leq \frac{t}{1+t^\alpha} \leq 1$ is trivial if you consider case $0 \leq t \leq 1$ and $t > 1$ separately. Now we can show that $\left|\frac{nx \sin x}{1+(nx)^\alpha}\right| \to 0$ for all $x \in [0, 2\pi]$ and use DCT.

   (ii) The inequality is trivial to show by comparing derivatives. After that one can use DCT to obtain the result.

2. We know that $f_n(x) \to H(x)$, where $H(x)$ is an appropriate step function. Using DCT it is clear that $N_1(f_n - f) \to 0$ and hence $\{f_n\}$ is Cauchy in N_1. However, we see that the limit is discontinuous.

3. Trivial using properties of the norm.

4. For $p = 1$ we did it in problem 4, HW3. If $1 < p < \infty$ we do exactly the same argument and use DCT. When $p = \infty$ we use the fact that $-C \leq f(x) \leq C$ in X (trivial modification is needed for a.e.) and then define

$$\phi_n(x) = \frac{m}{n} \text{ on } A_m^n = \{x \in X : m/n \leq f(x) < (m+1)/n\},$$

   where $n \in \mathbb{N}$, $-Cn \leq m \leq Cn$. It is clear that each f_n is simple and $|f_n(x) - f(x)| \leq \frac{1}{n}$.

5. We have that

$$\int |f|^p \mathrm{d}\mu = \sum_{n=1}^{\infty} \frac{\sqrt{n}^p}{n^2} = \sum_{n=1}^{\infty} n^{p/2-2}.$$

   Thus $f \in L_p$ if and only if $(p/2 - 2) < -1$ which happens if and only if $1 \leq p < 2$.

   For the second part we take $f(n) = n^{1/p_0}/\log(n)^2$. We then have that

$$\int |f|^p = \sum_{n=1}^{\infty} (n^{2-p/p_0} (\log n)^{2p})^{-1}.$$

   Thus if $p > p_0$ we have that $p/p_0 > 1$ and

$$\int |f|^p = \sum_{n=1}^{\infty} (n^{2-p/p_0} (\log n)^{2p})^{-1} = \infty$$

   and if $p < p_0$ then

$$\int |f|^p = \sum_{n=1}^{\infty} (n^{2-p/p_0} (\log n)^{2p})^{-1} < \infty.$$

Moreover for $p = p_0$

$$\int |f|^{p_0} = \sum_{n=1}^{\infty} (n(\log n)^{2p_0})^{-1} < \infty.$$

6. Firstly for $k \in \mathbb{N}$ let $F_k = \{x : |f(x)| < k\}$ and note that

$$\int |f| \mathrm{d}\mu = \lim_{k\to\infty} \int |f| \chi_{F_k} \mathrm{d}\mu. \tag{1}$$

If we let $\phi_k = \sum_{n=1}^{k} (n-1)\chi_{E_n}$ and $\psi_k = \sum_{n=1}^{k} n\chi_{E_n}$ then

$$\sum_{n=1}^{k} (n-1)\mu(E_n) = \int \phi_k \mathrm{d}\mu \le \int |f|\chi_{F_k} \mathrm{d}\mu \le \int \psi_k \mathrm{d}\mu = \sum_{n=1}^{k} n\mu(E_n).$$

Thus applying equation (1) we have that

$$\sum_{n=1}^{\infty} (n-1)\mu(E_n) \le \int |f| \mathrm{d}\mu \le \sum_{n=1}^{\infty} n\mu(E_n)$$

and the result follows. Extension to L^p is trivial.

7. Let q satisfy that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Since $f \in L_p$ and $1 \in L_q$ ($\mu(X) < \infty$) we can apply Corollary 6.10 from the notes to get that $f = f \cdot 1 \in L_r$ and

$$\|f\|_r \le \|1\|_q \|f\|_p = \mu(X)^{1/q} \|f\|_p = \mu(X)^{1/r - 1/p} \|f\|_p.$$

8. Since $f \in L_p$ with respect to counting measure we have that $\sum_{n=1}^{\infty} |f(n)|^p < \infty$. Thus there exists N such that for all $n \ge N$ $|f(n)| < 1$ and so $|f(n)|^p \ge |f(n)|^s$. Therefore

$$\int |f|^s = \sum_{n=1}^{\infty} |f(n)|^s \le \sum_{n=1}^{N} |f(n)|^s + \sum_{n=N+1}^{\infty} |f(n)|^p < \infty.$$

9. For $p = 2$ it is easy to integrate. For $p \ne 2$ we see that there is a blow up of the integral either near 0 or at ∞.

10. Fix $p_1 \le p \le p_2$ and α such that $\alpha/p_1 + (1-\alpha)/p_2 = 1/p$. We then have that $f^\alpha \in L_{p_1/\alpha}$ and $f^{1-\alpha} \in L_{p_2/\alpha}$. Thus we can apply Corollary 6.10 from the notes to see that $f = f^\alpha f^{1-\alpha} \in L_p$ and

$$\|f\|_p \le \|f^\alpha\|_{p_1} \|f^{1-\alpha}\|_{p_2}.$$

11. Use Holder inequality to show that $F(x) \le \left(\int_0^x f^p dm\right)^{1/p} x^{1/q}$. For $x \to \infty$ result follows from the fact that $f \in L^p$. For $x \to 0$ it follows from the fact that $\int_0^x f^p dm \to 0$.

12. Since $\|f\| > A$ we know that there exists $a > 0$ such that $\|f\| = A + a$. Let's argue by contradiction, assume that $\mu(\{x \in X : |f(x)| > A\}) = 0$. Then we know that $|f(x)| \leq A$ a.e $x \in X$ but then $\|f\| \leq A$ and we get a contradiction.

13. Since $g \in L_\infty$ we know that if we let

$$A = \{x : |g(x)| > \|g\|_\infty\}$$

then $\mu(A) = 0$. Thus

$$\int |fg|^p \mathrm{d}\mu = \int_{A^c} |fg|^p \leq \|g\|_\infty^p \int |f|^p \mathrm{d}\mu < \infty$$

and taking the pth root yields that

$$\|fg\|_p \leq \|g\|_\infty \|f\|_p.$$

14. (a) First suppose that $\mu(X) < \infty$ and $f \in L_\infty$. Thus there exists $K > 0$ and $A \in \mathbb{X}$ such that $\mu(A^c) = 0$ and for all $x \in A$ we have that $|f(x)| \leq K$. This means that

$$\int |f| \mathrm{d}\mu \leq \int_A |f| \mathrm{d}\mu + \int_{A^c} |f| \mathrm{d}\mu \leq K\mu(X) + 0 < \infty$$

and so $f \in L_1$. On the other hand if $\mu(X) = \infty$ then $1 \in L_\infty$ but $\int |1| \mathrm{d}\mu = \infty$ and so $1 \notin L_1$.

(b) Let $f \in L_\infty$. We know that

$$\mu(\{x : |f(x)| > \|f\|_\infty\}) = 0$$

and so

$$\left(\int |f|^p\right)^{1/p} \leq \|f\|_\infty.$$

On the other hand for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(\{x : |f| > \|f\|_\infty - \epsilon\}) > \delta.$$

Therefore

$$\left(\int |f|^p\right)^{1/p} \geq (\delta(\|f\|_\infty - \epsilon)^p)^{1/p} = \delta^{1/p}(\|f\|_\infty - \epsilon).$$

Thus for any $\epsilon > 0$ we have that

$$\liminf_{p \to \infty} \|f\|_p \geq \|f\|_\infty - \epsilon$$

and the proof is complete.

15. Use estimate on $\sin x$ in terms of a linear function.