MTI Exercises 6: Solutions

1. We can prove these results anologously to the corresponding results for measures. For part a) We let $E_0 = \emptyset$ and note that the sets $A_n = E_n \backslash E_{n-1}$ are disjoint. We have that

$$\nu(E_n) = \nu(E_{n-1}) + \nu(E_n \backslash E_{n-1})$$

and so

$$\nu(E_n \backslash E_{n-1}) = \nu(E_n) - \nu(E_{n-1}).$$

Thus since $E = \cup_{n=1}^{\infty} E_n = \cup_{n=1}^{\infty} A_n$ and the sets A_n are disjoint.

$$\nu \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \nu(E_n) - \nu(E_{n-1}) = \lim_{n\to\infty} \nu(E_n).$$

For part b) we let $B_0 = X$ and $B_n = F_{n-1} \backslash F_n$ for $n \in \mathbb{N}$. Again we can see these sets are disjoint, that $\nu(B_n) = \nu(F_{n-1} - \nu(F_n))$ and we will have that $F^c = \cup_{n=1}^{\infty} B_n$. Thus

$$\nu(F^c) = \sum_{n=1}^{\infty} \nu(B_n) = \nu(X) - \lim_{n\to\infty} \nu(B_n)$$

and thus $\nu(F) = \nu(X) - \nu(F^c) = \lim_{n\to\infty} \nu(B_n)$.

2. Let N, P be the Hahn decomposition for ν. We have that

$$\nu^+(E) = \nu(E \cap P) \leq \sup\{\nu(F) : F \subset E\}$$

On the other hand if $F \subset E$ then

$$\nu(F) = \nu(F \cap P) + \nu(F \cap N)$$

and since $\nu(F \cap N) \leq 0$ we have

$$\nu(F) \leq \nu(F \cap P) = \nu^+(F) \leq \nu^+(E).$$

Now take the supremum over all measurable sets F to get

$$\nu^+(E) = \{\nu(F) : F \subset E\}.$$

3. First suppose that $\mu(E \cap \{x \in X : f(x) \neq 0\}) = 0$ and let $B = \{x \in X : f(x) \neq 0$. Then for $A \subset E$ we have that

$$\nu(A) = \int_A f \mathrm{d}\mu = \int_{\mathrm{B\cap A}} \mathrm{fd}\mu + \int_{\mathrm{B^c\cap A}} \mathrm{fd}\mu = 0 + 0$$

since $\mu(B \cap A) = 0$ and $f(x) = 0$ for all $x \in B^c$. Thus E is a null set.

On the other hand suppose that E is a null set. Then consider $B^+ = E \cap \{x \in X : f(x) > 0\}$. We have that $\int_{B+} f \mathrm{d}\mu = 0$ since E is a null set. Thus since f is non-negative on E we must have that $f(x) = 0$ for μ-almost all $x \in B^+$ but since no $x \in E$ satisfy this we must have that $\mu(B^+) = 0$. Now consider $B^- = E \cap \{x \in X : f(x) < 0\}$. We have that $\int_{B^-} -f \mathrm{d}\mu = 0$ and thus we must have that $\mu(B^-) = 0$. Putting this together gives that

$$0 = \mu(B^- \cup B^+) = \mu(E \cap \{x \in X : f(x) \neq 0\}) = 0.$$

4. We use the definition of ν to find its Hahn decomposition directly. Let

$$P = \{x \in X : f(x) > 0\} \text{ and } N = P^c.$$

If $E \in X$ then $\nu(E \cap P) = \int_{E \cap P} f \mathrm{d}\mu \geq 0$ and $\nu(E \cap N) = \int_{E \cap N} f \mathrm{d}\nu \leq 0$. So P and N give a Hahn decomposition for ν. Moreover for $x \in P$ we have that $f^-(x) = 0$ and for $x \in \mathbb{N}$ we have that $f^+(x) = 0$. Therefore for all $A \in \mathbb{X}$

$$\nu^+(A) = \int_A f^+ \mathrm{d}\mu \text{ and } \nu^-(A) = \int f^- \mathrm{d}\mu.$$

5. We have that $xe^{-x^2} > 0$ if and only if $x > 0$. So we take $P = (0, \infty)$ and $N = (-\infty, 0]$ (it does not matter which set we choose to put 0 in).

6. Let $\phi \in M^+$ be a simple function, written as $\sum_{i=1}^n c_i \chi_{A_i}$ in standard form. We have that by the Radon-Nikodým Theorem

$$\int \phi \mathrm{d}\nu = \sum_{i=1}^n c_i \nu(A_i) = \sum_{i=1}^n c_i \int_{A_i} f \mathrm{d}\mu = \int \phi f \mathrm{d}\mu.$$

So the result holds for all non-negative simple functions. We now let $g \in M^+$ and ϕ_n a sequence of non-negative functions which converge monotonically to g. By applying the monotone convergence theorem twice (to ϕ_n and to $f\phi_n$) we get that

$$\int g f \mathrm{d}\mu = \lim_{n \to \infty} \int \phi_n f \mathrm{d}\mu = \lim_{n \to \infty} \int \phi_n \mathrm{d}\nu = \int f \mathrm{d}\nu.$$

7. For the first part we let $A \in \mathbb{X}$ with $\mu(A) = 0$. We then know that $\lambda(A) = 0$ since λ is absolutely continuous with respect to μ and thus $\nu(A) = 0$ since ν is absolutely continuous with respect to λ.

For the second part let $h = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, $f = \frac{\mathrm{d}\nu}{\mathrm{d}\lambda}$ and $g = \frac{\mathrm{d}\lambda}{\mathrm{d}\mu}$. Let $A \in \mathbb{X}$ and use the result from question 5 and Radon-Nikodým to get that

$$\int_A h \mathrm{d}\mu = \nu(A) = \int_A f \mathrm{d}\lambda = \int_A f g \mathrm{d}\mu.$$

This holds for all $A \in \mathbb{X}$ and so in particular holds for the measurable set $\{x \in X : h(x) \neq fg(x)\}$. Thus

$$\mu(\{x \in X : h(x) \neq fg(x)\} = 0$$

which completes the proof.

8. For $E \in \mathbb{B}$ we define $\nu_2(E) = \nu(E \cap (-\infty, 0])$ and $\nu_1(E) = (E \cap (0, \infty))$. We can see straight away that $\nu = \nu_1 + \nu_2$. We have that $\mu((0, \infty)) = 0$ and $\nu_1((-\infty, 0]) = 0$ so μ and ν_1 are mutually singular. On the other hand if $\mu(A) = 0$ for $A \in \mathbb{B}$ then since $g(x) > 0$ for all $x \leq 0$ we know that $\lambda(A \cap (-\infty, 0)) = 0$. This means that

$$\nu_2(A) = \int_{A \cap (-\infty, 0)} f \mathrm{d}\lambda = 0.$$

Thus ν_2 is absolutely continuous with respect to μ and ν_1 and ν_2 is the Lebesgue decomposition with respect to μ for ν.

9. First suppose that f is non-negative. Let ν be the measure on $(X, \mathbb{X}_0)$ defined by

$$\nu(A) = \int_A f \mathrm{d}\mu.$$

We know that ν is absolutely continuous with respect to μ on $(X, \mathbb{X}_0)$ and since f is integrable ν is finite. We can define $\nu_{\mathbb{X}}, \mu_{\mathbb{X}} : \mathbb{X} \to \mathbb{R}$ by defining that for $A \in \mathbb{X}$ $\mu_{\mathbb{X}}(A) = \mu(A)$ and $\nu_{\mathbb{X}}(A) = \nu(A)$. It immediately follows that since μ and ν are measures on $(X, \mathbb{X}_0)$ with $\nu << \mu$ that $\mu_{\mathbb{X}}$ and $\nu_{\mathbb{X}}$ are measures on $(X, \mathbb{X})$ with $\nu_{\mathbb{X}}$ absolutely continuous with respect to $\nu_{\mathbb{X}}$. Therefore by the Radon-Nikodym theorem we can find a nonnegative function $g \in L(X, \mathbb{X}, \mu_{\mathbb{X}})$ such that for each $A \in \mathbb{X}$

$$\int_A g \mathrm{d}\mu_X = \nu_{\mathbb{X}}(A) = \nu(A) = \int_A f \mathrm{d}\mu$$

and thus since g must also be $(X, \mathbb{X}_0)$ measurable we have

$$\int_A g \mathrm{d}\mu = \int f \mathrm{d}\mu.$$

To complete the result we consider a general $f \in L(X, \mathbb{X}_0, \mu)$ and write $f = f^+ - f^-$ and apply the above argument to f^+ and f^-. Note that f may not be measurable in $(X, \mathbb{X})$ so we cannot just take $f = g$.