**Measure Theory, MA 359** **Handout 1**

**Valeriy Slastikov** **Autumn, 2005**

# 1 Measure theory

## 1.1 General construction of Lebesgue measure

In this section we will do the general construction of $\sigma$-additive complete measure by extending initial $\sigma$-additive measure on a semi-ring to a measure on $\sigma$-algebra generated by this semi-ring and then completing this measure by adding to the $\sigma$-algebra all the null sets. This section provides you with **the essentials** of the construction and make some parallels with the construction on the plane.

Throughout these section we will deal with some *collection of sets* whose elements are subsets of some fixed **abstract** set $X$. It is not necessary to assume **any** topology on $X$ but for simplicity you may imagine $X = \mathbb{R}^n$.

We start with some important definitions:

**Definition 1.1** *A nonempty collection of sets $\mathfrak{S}$ is a semi-ring if*

1. *Empty set $\varnothing \in \mathfrak{S}$;*
2. *If $A \in \mathfrak{S}, B \in \mathfrak{S}$ then $A \cap B \in \mathfrak{S}$;*
3. *If $A \in \mathfrak{S}, A \supset A_1 \in \mathfrak{S}$ then $A = \cup_{k=1}^n A_k$, where $A_k \in \mathfrak{S}$ for all $1 \le k \le n$ and $A_k$ are disjoint sets.*

*If the set $X \in \mathfrak{S}$ then $\mathfrak{S}$ is called semi-algebra, the set $X$ is called a unit of the collection of sets $\mathfrak{S}$.*

**Example 1.1** *The collection $\mathfrak{S}$ of intervals $[a, b)$ for all $a, b \in \mathbb{R}$ form a semi-ring since*

1. *empty set $\varnothing = [a, a) \in \mathfrak{S}$;*
2. *if $A \in \mathfrak{S}$ and $B \in \mathfrak{S}$ then $A = [a, b)$ and $B = [c, d)$. Obviously the intersection $A \cap B$ is either empty set or an interval. Therefore $A \cap B \in \mathfrak{S}$;*
3. *if $A \in \mathfrak{S}, A \supset A_1 \in \mathfrak{S}$ then $A = [a, b)$ and $A_1 = [c, d)$, where $c \ge a$ and $d \le b$. Obviously you may find two intervals $[a, c) \in \mathfrak{S}$ and $[d, b) \in \mathfrak{S}$ such that $[a, b) = [a, c) \cup [c, d) \cup [d, b)$.*

*Note that $\mathfrak{S}$ is not a semi-algebra since all those intervals are subsets of $\mathbb{R}$ but $\mathbb{R}$ can not be represented as an interval of form $[a,b)$ ($-\infty \notin \mathbb{R}$, but $\mathbb{R}$ can be represented as a countable union of such intervals).*

**Exercise 1.1** *Show that the collection $\mathfrak{S}$ of intervals $[a,b)$ for all $a,b \in [0,1]$ form a semi-algebra*

**Exercise 1.2** *Let $\mathfrak{S}$ be the collection of rectangles in the plane $(x,y)$ defined by one of the inequalities of the form*

$$a \le x \le b, \quad a < x \le b, \quad a \le x < b, \quad a < x < b$$

*and one of the inequalities of the form*

$$c \le y \le d, \quad c < y \le d, \quad c \le y < d, \quad c < y < d.$$

*Show that*

- *if $a,b,c,d$ are arbitrary numbers in $\mathbb{R}$ then $\mathfrak{S}$ is a semi-ring;*
- *if $a,b,c,d$ are arbitrary numbers in $[0,1]$ then $\mathfrak{S}$ is a semi-algebra.*

**Definition 1.2** *A nonempty collection of sets $\mathfrak{R}$ is a ring if*

1. *Empty set $\varnothing \in \mathfrak{R}$;*
2. *If $A \in \mathfrak{R}, B \in \mathfrak{R}$ then $A \cap B \in \mathfrak{R}$, $A \cup B \in \mathfrak{R}$, and $A \setminus B \in \mathfrak{R}$.*

*If the set $X \in \mathfrak{R}$ then $\mathfrak{R}$ is called an algebra.*

**Exercise 1.3** *We call a set $A \subset \mathbb{R}^2$* **elementary** *if it can be written, in at least one way, as a finite union of disjoint rectangles from exercise 1.2. Show that the collection of elementary sets $\mathfrak{R}$ form a ring.*

**Definition 1.3** *A set function $\mu(A)$ defined on a collection of sets $\mathcal{S}_\mu$ is a measure if*

1. *Its domain of definition $\mathcal{S}_\mu$ is a semi-ring;*

2. $\mu(A) \geq 0$ *for all* $A \in \mathcal{S}_\mu$*;*

3. *If* $A_1, A_2 \in \mathcal{S}_\mu$ *are disjoint sets and* $\mathcal{S}_\mu \ni A = A_1 \cup A_2$ *then*

$$\mu(A) = \mu(A_1) + \mu(A_2)$$

Note that $\mu(\varnothing) = 0$ since $\mu(\varnothing) = \mu(\varnothing) + \mu(\varnothing)$

**Example 1.2** *If we define a set function* $m$ *on a semi-ring of rectangles* $\mathfrak{S}$ *from exercise 1.2 like:*

1. $m(\varnothing) = 0$*;*

2. *If* $R \in \mathfrak{S}$ *is a nonempty rectangle (closed, open or half open) defined by the numbers* $a, b, c, d$ *then* $m(R) = (b - a) * (d - c)$

*It is easy to check that* $m$ *is a measure on* $\mathfrak{S}$*.*

The following lemmas are going to be used in the notes. The proof is not difficult and is left as an exercise.

**Lemma 1.1** *The intersection* $\mathfrak{R} = \cap_\alpha \mathfrak{R}_\alpha$ *of an arbitrary number of rings is a ring.*

**Lemma 1.2** *If* $\mathfrak{S}$ *is an arbitrary nonempty collection of sets there exists precisely one ring* $\mathfrak{R}(\mathfrak{S})$ *containing* $\mathfrak{S}$ *and contained in every ring* $\mathfrak{R}$ *containing* $\mathfrak{S}$*. This ring* $\mathfrak{R}(\mathfrak{S})$ *is called the minimal ring over collection* $\mathfrak{S}$ *(or the ring generated by* $\mathfrak{S}$*).*

**Lemma 1.3** *If* $\mathfrak{S}$ *is a semi-ring then* $\mathfrak{R}(\mathfrak{S})$ *coincides with the collection of sets* $A$ *that admit a finite partition*

$$A = \cup_{k=1}^{n} A_k, \quad A_k \in \mathfrak{S}, \ A_i \cap A_j = \varnothing.$$

**Definition 1.4** *A measure* $\mu$ *is an extension of measure* $m$ *if domain of definition* $\mathfrak{S}_\mu$ *of measure* $\mu$ *contains domain of definition* $\mathfrak{S}_m$ *of measure* $m$ *(*$\mathfrak{S}_m \subset \mathfrak{S}_\mu$*) and*

$$\mu(A) = m(A) \text{ for all } A \in \mathfrak{S}_m.$$

**Exercise 1.4** *Take* $X = [0,1] \times [0,1]$ *and consider the collection* $\mathfrak{S}$ *of all rectangles from exercise 1.2 that are subsets of* $X$*. Define a measure* $m$ *on* $\mathfrak{S}$ *like in example 1.2. Show that it is possible to extend this measure* $m$ *to a measure* $m'$ *on the collection* $\mathfrak{R}$ *of elementary sets from exercise 1.3 that are subsets of* $X$*.*

Note that in the above exercise the domain of definition of $m'$ is actually a minimal algebra $\mathfrak{R}$ containing semi-algebra $\mathfrak{S}$ – domain of definition of $m$, we write it as $\mathfrak{R}_{m'} = \mathfrak{R}(\mathfrak{S}_m)$ (you can also say that $\mathfrak{R}_{m'}$ is generated by $\mathfrak{S}_m$). And therefore $m'$ is actually an extension of $m$ from semi-algebra to an algebra. This can be generalized into the following theorem.

**Theorem 1.5** *Every measure $m(A)$ whose domain of definition $\mathfrak{S}_m$ is a semi-ring has unique extension $\mu(A)$ whose domain of definition $\mathfrak{R}_\mu$ is a ring generated by $\mathfrak{S}_m$, i.e. $\mathfrak{R}_\mu = \mathfrak{R}(\mathfrak{S}_m)$.*

**Proof** For every set $A \in \mathfrak{R}_\mu$ there exists a partition

$$A = \cup_{i=1}^n B_i, \quad \text{where } B_i \in \mathfrak{S}_m \tag{1}$$

We define

$$\mu(A) = \sum_{i=1}^n m(B_i) \tag{2}$$

The value of $\mu(A)$ is independent of the partition (1). To see this we assume that there are two partitions of $A$: $A = \cup_{i=1}^n B_i = \cup_{j=1}^k Q_j$, where $B_i, Q_j \in \mathfrak{S}_m$ and $B_i \cap B_j = \varnothing$, $Q_i \cap Q_j = \varnothing$ for $i \neq j$. Since $B_i \cap Q_j \in \mathfrak{S}_m$ and by additivity property of a measure $m$ we have

$$\sum_{i=1}^n m(B_i) = \sum_{i=1}^n \sum_{j=1}^k m(B_i \cap Q_j) = \sum_{j=1}^k \sum_{i=1}^n m(B_i \cap Q_j) = \sum_{j=1}^k m(Q_j)$$

So $\mu(A)$ is well defined. Obviously $\mu$ is nonnegative and additive. This takes care of existence part.

Let us show its uniqueness. Suppose there are two measures $\mu$ and $\tilde{\mu}$ that are extensions of $m$. For any $A \in \mathfrak{R}(S_m)$ we have $A = \cup_{i=1}^n B_i$, where $B_i \in \mathfrak{S}_m$. By definition of extension $\mu(B_i) = \tilde{\mu}(B_i) = m(B_i)$, so using additivity of a measure:

$$\tilde{\mu}(A) = \sum_{i=1}^n \tilde{\mu}(B_i) = \sum_{i=1}^n \mu(B_i) = \mu(A)$$

So $\mu$ and $\tilde{\mu}$ coincide. Theorem is proved.

Note that we proved not only existence of an extension but also its uniqueness. It allows us to claim that measure $m'$ defined on the collection of elementary sets is *the only possible* extension of the measure $m$ defined on the collection of all rectangles.

**Definition 1.6** *A measure $\mu$ is called semiadditive (or countably subadditive) if for any $A, A_1, A_2, ... \in \mathfrak{S}_\mu$ such that $A \subset \cup_{n=1}^{\infty} A_n$*

$$\mu(A) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

**Definition 1.7** *A measure $\mu$ is called $\sigma$-additive if for any $A, A_1, A_2, ... \in \mathfrak{S}_\mu$ such that $A = \cup_{n=1}^{\infty} A_n$ and $\{A_n\}$ are disjoint*

$$\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$$

**Theorem 1.8** *If a measure $\mu$ defined on a ring $\mathfrak{R}_\mu$ is semiadditive then it is $\sigma$-additive.*

**Proof** Let $A_n, A \in \mathfrak{R}_\mu$ for all $n$ and $A = \cup_{n=1}^{\infty} A_n$. For any $N \in \mathbb{N}$ we have $\cup_{n=1}^{N} A_n \subset A$, $\mu$ is a measure and hence it is additive. Therefore

$$\mu(\cup_{n=1}^{N} A_n) = \sum_{n=1}^{N} \mu(A_n) \leq \mu(A)$$

Letting $N \to \infty$ we obtain

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu(A)$$

On the other hand, by semiadditivity we have

$$\sum_{n=1}^{\infty} \mu(A_n) \geq \mu(A)$$

The theorem is proved.

Note that here we required the domain of definition of $\mu$ to be a ring since otherwise it is not clear if $\mu(\cup_{n=1}^{N} A_n)$ is defined.

**Exercise 1.5** *Prove that measure $m'$ from exercise 1.4, defined on the collection $\mathfrak{R}_{m'}$ of elementary sets is $\sigma$-additive.*

**Exercise 1.6** *Prove that measure $m$ from example 1.2 , defined on the collection $\mathfrak{S}_m$ of rectangles is semiadditive. Is it $\sigma$-additive?*

**Theorem 1.9** *If a measure $m$ defined on a semi-ring $\mathfrak{S}_m$ is $\sigma$-additive then its extension $\mu$ to a minimal ring $\mathfrak{R}(\mathfrak{S}_m)$ is $\sigma$-additive.*

**Proof** Assume that $A \in \mathfrak{R}(\mathfrak{S}_m)$ and $B_n \in \mathfrak{R}(\mathfrak{S}_m)$ for $n = 1, 2, ...$ are such that $A = \cup_{n=1}^{\infty} B_n$ and $B_n$ are disjoint. Then there exist disjoint sets $A_i \in \mathfrak{S}_m$, and disjoint sets $B_{nj} \in \mathfrak{S}_m$ such that

$$A = \cup_i A_i \quad \text{and} \quad B_n = \cup_j B_{nj},$$

where the unions are finite.

Let $C_{nji} = B_{nj} \cap A_i$. Obviously $C_{nji}$ are disjoint and

$$A_i = \cup_n \cup_j C_{nji} \quad \text{and} \quad B_{nj} = \cup_i C_{nji}.$$

By complete additivity of $m$ on $\mathfrak{S}_m$ we have

$$m(A_i) = \sum_n \sum_j m(C_{nji}) \quad \text{and} \quad m(B_{nj}) = \sum_i m(C_{nji}).$$

By definition of $\mu$ on $\mathfrak{R}(\mathfrak{S}_m)$ we have

$$\mu(A) = \sum_i m(A_i) \quad \text{and} \quad \mu(B_n) = \sum_j m(B_{nj})$$

Since sums in $i$ and $j$ are finite and the series in $n$ converges, it is easy to see from the above equalities that

$$\mu(A) = \sum_i \sum_n \sum_j m(C_{nji}) = \sum_n \sum_j \sum_i m(C_{nji}) = \sum_n \mu(B_n).$$

Theorem is proved.

**Definition 1.10** *A nonempty collection of sets $\mathfrak{P}$ is a $\sigma$-ring if*

1. *Empty set $\varnothing \in \mathfrak{R}$;*
2. *If $A_n \in \mathfrak{P}$, $n = 1, 2, ...$ then $\cap_{n=1}^{\infty} A_n \in \mathfrak{P}$, $\cup_{n=1}^{\infty} A_n \in \mathfrak{P}$;*
3. *If $A, B \in \mathfrak{P}$ then $A \setminus B \in \mathfrak{P}$.*

*If the set $X \in \mathfrak{P}$ then $\mathfrak{P}$ is called an $\sigma$-algebra.*

**Lemma 1.4** *The intersection $\mathfrak{P} = \cap_\alpha \mathfrak{P}_\alpha$ of an arbitrary number of $\sigma$-rings is a $\sigma$-ring*

**Lemma 1.5** *If $\mathfrak{S}$ is an arbitrary nonempty collection of sets there exists precisely one $\sigma$-ring $\mathfrak{P}(\mathfrak{S})$ containing $\mathfrak{S}$ and contained in every $\sigma$-ring $\mathfrak{P}$ containing $\mathfrak{S}$. This $\sigma$-ring $\mathfrak{P}(\mathfrak{S})$ is called the minimal $\sigma$-ring over collection $\mathfrak{S}$ (or the $\sigma$-ring generated by $\mathfrak{S}$).*

**Definition 1.11** *A measure $\mu$ is called finite if for every $A \in \mathfrak{S}_\mu$ $\mu(A) < \infty$. A measure $\mu$ is called $\sigma$-finite if for every $A \in \mathfrak{S}_\mu$ there exists a sequence of sets $\{A_n\} \subset \mathfrak{S}_\mu$ such that $A \subset \cup_n A_n$ and $\mu(A_n) < \infty$.*

**Theorem 1.12** *Every $\sigma$-additive $\sigma$-finite measure $m(A)$ whose domain of definition $\mathfrak{R}_m$ is a ring has unique extension $\mu(A)$ whose domain of definition $\mathfrak{P}(\mathfrak{R}_m)$ is a minimal $\sigma$-ring generated by $\mathfrak{R}_m$ and $\mu$ is $\sigma$-additive and $\sigma$-finite.*

**Proof** We prove this theorem for a finite measure defined on an algebra. The existence follows from the the theorems in section 1.2 and uniqueness is left as an exercise.

**Remark** Theorems 1.5, 1.9, 1.12 tell us that if we have a $\sigma$-additive $\sigma$-finite measure $m$ on a semi-ring $\mathfrak{S}_m$, there is **unique** extension $\mu$ of this measure to the minimal $\sigma$-ring $\mathfrak{P}(\mathfrak{S}_m)$ and moreover this extension $\mu$ is $\sigma$-additive and $\sigma$-finite. Therefore one can always start defining the $\sigma$-additive $\sigma$-finite measure directly on a $\sigma$-ring, not on a semi-ring. You can find this approach in many textbooks.

**Remark** Theorem 1.12 tells you that one can extend measure $m'$ defined on the ring $\mathfrak{R}_{m'}$ of elementary sets in $\mathbb{R}^2$ to the unique $\sigma$-additive measure $\mu$ defined on a minimal $\sigma$-ring $\mathfrak{P}(\mathfrak{R}_{m'})$. It is easy to see that $\mathfrak{P}(\mathfrak{R}_{m'})$ coincides with $\sigma$-algebra of all open sets in $\mathbb{R}^2$ (or Borel algebra).
Thus, starting from a measure $m$ on rectangles one can construct unique $\sigma$-additive measure $\mu$ on Borel algebra.

**Theorem 1.13** *If $A_1 \supset A_2 \supset \ldots$ is a monotone decreasing sequence of sets in $\sigma$-ring $\mathfrak{P}$, $\mu$ is a $\sigma$-additive measure on $\mathfrak{P}$, $A = \cap_n A_n$ and $\mu(A_1) < \infty$ then*

$$\mu(A) = \lim_{n \to \infty} \mu(A_n)$$

**Proof** It is enough to consider the case $A = \varnothing$ since the general case reduces to this on replacing $A_n$ by $A_n \backslash A$. Now

$$A_1 = (A_1 \backslash A_2) \cup (A_2 \backslash A_3) \cup \ldots, \quad A_n = (A_n \backslash A_{n+1}) \cup (A_{n+1} \backslash A_{n+2}) \cup \ldots$$

and by $\sigma$-additivity of $\mu$ we have

$$\mu(A_1) = \sum_{n=1}^{\infty} \mu(A_n \backslash A_{n+1}), \quad \mu(A_k) = \sum_{n=k}^{\infty} \mu(A_n \backslash A_{n+1}).$$

Since the $\mu(A_1) < \infty$ we have $\mu(A_k) \to 0$ as $k \to \infty$. Theorem is proved.

**Definition 1.14** *Let $\mu$ be a $\sigma$-additive measure defined on Borel $\sigma$-algebra of all open sets in $\mathbb{R}^n$. The $\mu$ is called* **Borel** *measure.*

**Definition 1.15** *A measure $\mu$ defined on a semi-ring $\mathfrak{S}_\mu$ is called complete if $A \in \mathfrak{S}_\mu$, $B \subset A$ and $\mu(A) = 0$ imply that $B \in \mathfrak{S}_\mu$ (and then $\mu(B) = 0$).*

**Theorem 1.16** *If $\mu$ is a $\sigma$-additive measure on a $\sigma$-ring $\mathfrak{P}$ then the class $\mathfrak{M}$ of all sets $A \subset X$ of the form*

$$A = B \cup N, \tag{3}$$

*where $B \in \mathfrak{P}$ and $N \subset E \in \mathfrak{P}$ such that $\mu(E) = 0$, is a $\sigma$-ring and the set function $\bar{\mu}$ defined by*

$$\bar{\mu}(A) = \mu(B),$$

*for all $A \in \mathfrak{M}$ and $B \in \mathfrak{P}$ related like in (3) is a complete $\sigma$-additive measure on $\mathfrak{M}$.*
*The measure $\bar{\mu}$ is called completion of $\mu$.*

**Proof** Let us show that $\mathfrak{M}$ is a $\sigma$-ring.

1. Obviously $\varnothing \in \mathfrak{M}$, since $\varnothing$ can be represented as $\varnothing = \varnothing \cup \varnothing$.

2. If $A_i \in \mathfrak{M}$ for $i = 1, 2, ...$ then

   $$A_i = B_i \cup N_i,$$

   where $B_i \in \mathfrak{P}$ and $N_i \subset E_i \in \mathfrak{P}$ such that $\mu(E_i) = 0$. We want to check if $A = \cup_i A_i \in \mathfrak{M}$: by definition $A = \cup_i B_i \cup \cup_i N_i$. Since $\mathfrak{P}$ is a *sigma*-ring $B = \cup_i B_i \in \mathfrak{P}$, $N = \cup_i N_i \subset \cup_i E_i \in \mathfrak{P}$ and $\mu(\cup_i E_i) \leq \sum_i \mu(E_i) = 0$. Therefore $A \in \mathfrak{M}$.

   Now we want to check if $A = \cap_i A_i \in \mathfrak{M}$: by definition $A = \cap_i B_i \cup \cap_i N_i$. Since $\mathfrak{P}$ is a $\sigma$-ring $B = \cap_i B_i \in \mathfrak{P}$, $N = \cap_i N_i \subset E_1 \in \mathfrak{P}$ and $\mu(E_1) = 0$. Therefore $A \in \mathfrak{M}$.

3. Let $A_1, A_2 \in \mathfrak{M}$ then $A_1 \backslash A_2 = (B_1 \backslash A_2) \cup (N_1 \backslash A_2) = ((B_1 \backslash B_2) \backslash N_2) \cup (N_1 \backslash A_2)$. Obviously $B = (B_1 \backslash B_2) \in \mathfrak{P}$ and $B \backslash N_2 = (B \backslash E_2) \cup (E_2 \cap (B \backslash N_2))$, since $N_2 \subset E_2$. We obtain

$$A_1 \backslash A_2 = (B \backslash E_2) \cup ((E_2 \cap (B \backslash N_2)) \cup (N_1 \backslash A_2)),$$

where $B \backslash E_2 \in \mathfrak{P}$ and $(E_2 \cap (B \backslash N_2)) \cup (N_1 \backslash A_2) \subset E_2 \cup E_1$ with $\mu(E_1 \cup E_2) = 0$. So $A_1 \backslash A_2 \in \mathfrak{M}$.

We showed that $\mathfrak{M}$ is a $\sigma$-ring. It is easy to prove that if $\mathfrak{P}$ is a $\sigma$-algebra then $\mathfrak{M}$ is a $\sigma$-algebra.

Now we have to check that the set function $\bar{\mu}$ is well defined, i.e. we have to show that if $A_1 \cup N_1 = A_2 \cup N_2$ then $\bar{\mu}(A_1 \cup N_1) = \bar{\mu}(A_2 \cup N_2)$. By definition

$$\bar{\mu}(A_1 \cup N_1) = \mu(A_1) \text{ and } \bar{\mu}(A_2 \cup N_2) = \mu(A_2),$$

so we have to prove that $\mu(A_1) = \mu(A_2)$. Using the fact $A_1 \cup N_1 = A_2 \cup N_2$ we obtain

$$\begin{aligned} \mu(A_1) = \mu(A_1 \cup E_1) \leq \mu(A_2 \cup E_1 \cup E_2) = \mu(A_2) \\ \mu(A_2) = \mu(A_2 \cup E_2) \leq \mu(A_1 \cup E_1 \cup E_2) = \mu(A_1). \end{aligned}$$

Therefore $\bar{\mu}$ is well defined.

Now we have to show that $\bar{\mu}$ is $\sigma$-additive measure.

1. It is obvious $\bar{\mu}(\varnothing) = 0$.

2. It is obvious $\bar{\mu}(A) \geq 0$ for any $A \in \mathfrak{M}$.

3. If $A_i \in \mathfrak{M}$ for $i = 1, 2, ...,$ $A_i \cap A_j = \varnothing$ and $A = \cup_i A_i$ then

$$\bar{\mu}(A) = \sum_i \bar{\mu}(A_i).$$

   To show this we notice that $\bar{\mu}(A) = \mu(\cup_i B_i)$ and since $A_i$'s are disjoint the same is true for $B_i$'s (recall that $A_i = B_i \cup N_i$). Now by $\sigma$-additivity of $\mu$ we obtain

$$\bar{\mu}(A) = \mu(\cup_i B_i) = \sum_i \mu(B_i) = \sum_i \bar{\mu}(A_i).$$

Therefore $\bar{\mu}$ is a $\sigma$-additive measure. It is easy to check that it is complete: if $A \in \mathfrak{M}$ and $B \subset A$, and $\bar{\mu}(A) = 0$ then $A \subset E$, where $E \in \mathfrak{P}$ and $\mu(E) = 0$. But then $B = \varnothing \cup B$, $B \subset E$ hence $B \in \mathfrak{M}$. The theorem is proved.

**Definition 1.17** *The completion of translation invariant Borel measure in* $\mathbb{R}^n$ *is called* **Lebesgue** *measure.*

**Remark** For simplicity, we define Borel and Lebesgue measures in $\mathbb{R}^n$. These definitions may be transferred to some topological spaces.

## 1.2 The extension of a measure on a semi-algebra using outer measure

Here we are going to introduce the second approach to the construction of complete $\sigma$-additive measure. Let $m$ be a $\sigma$-additive measure defined on a semi-algebra $\mathfrak{S}_m$ with a unit $X$.

**Definition 1.18** *For any set* $A \subset X$ *we define the outer measure*

$$\mu^*(A) = \inf_{A \subset \cup_n B_n} \sum_n m(B_n),$$

*where infimum is taken over all coverings of* $A$ *by countable collections of sets* $B_n \in \mathfrak{S}_m$.

Let $m'$ be an extension of $m$ to an algebra $\mathfrak{R}(\mathfrak{S}_m)$ (it exists by theorem 1.5). Then we may give an equivalent definition of the outer measure $\mu^*$.

**Definition 1.19** *For any set* $A \subset X$ *we define the outer measure*

$$\mu^*(A) = \inf_{A \subset \cup_n B_n} \sum_n m'(B_n),$$

*where infimum is taken over all coverings of* $A$ *by countable collections of sets* $B_n \in \mathfrak{R}(\mathfrak{S}_m)$.

Let $\mu_1$ be an extension of $m'$ to a $\sigma$-algebra $\mathfrak{P}(\mathfrak{S}_m)$ (it exists by theorem 1.12). Then we may give yet another equivalent definition of the outer measure $\mu^*$ (we are not going to use this one).

**Definition 1.20** *For any set* $A \subset X$ *we define the outer measure*

$$\mu^*(A) = \inf_{A \subset B} \mu_1(B),$$

*where infimum is taken over all coverings of* $A$ *by sets* $B \in \mathfrak{P}(\mathfrak{S}_m)$.

It is easy to see that these three definitions are equivalent.

Let's prove some properties of $\mu^*$.

**Theorem 1.21** *If $A \subset \cup_n A_n$ for some countable collection of sets $A_n$ then*

$$\mu^*(A) \le \sum_n \mu^*(A_n)$$

**Proof** By definition of $\mu^*$ for all $n$ and any $\epsilon > 0$ there exists a countable collection of sets $\{B_n^k\} \subset \mathfrak{S}_m$ such that $A_n \subset \cup_k B_n^k$ and

$$\sum_k m(B_n^k) \le \mu^*(A_n) + \frac{\epsilon}{2^n}.$$

Then $A \subset \cup_n \cup_k B_n^k$ and

$$\mu^*(A) \le \sum_n \sum_k m(B_n^k) \le \sum_n \mu^*(A_n) + \epsilon.$$

Taking $\epsilon \to 0$ we get the result.

**Lemma 1.6** *For any $A, B \subset X$ we have*

$$|\mu^*(A) - \mu^*(B)| \le \mu^*(A \Delta B).$$

**Proof** Since $A \subset B \cup (A\Delta B)$ and $B \subset A \cup (A\Delta B)$ we have

$$\mu^*(A) \le \mu^*(B) + \mu^*(A\Delta B), \quad \mu^*(B) \le \mu^*(A) + \mu^*(A\Delta B).$$

This implies $|\mu^*(A) - \mu^*(B)| \le \mu^*(A\Delta B)$. The lemma is proved.

It seems that $\mu^*$ is a very "good" set function: we can measure **any** subset of $X$ with it. But the "bad" thing about it is that $\mu^*$ is not additive (i.e. $\mu^*(A \cup B) \ne \mu^*(A) + \mu^*(B)$ if $A \cap B = \varnothing$) and hence it is not a measure in the usual sense. To see this in one particular case when $X = \mathbb{R}$ we construct **Vitali** set and use this construction to show non-additivity of $\mu^*$.

**Example 1.3** *(Vitali set) We define the following relation: for $x, y \in \mathbb{R}$ we say $x \sim y$ if and only if $x - y \in \mathbb{Q}$ ($\mathbb{Q}$ is the set of rational numbers). It is easy to check that*

1. *$x \sim x$;*
2. *$x \sim y \Rightarrow y \sim x$;*
3. *$x \sim y$ and $y \sim z \Rightarrow x \sim z$;*

*and hence $\sim$ is equivalence relation and $\mathbb{R}$ can be split into disjoint equivalence classes. We define set $E$ as a set containing* **exactly one** *representative from each equivalence class. Since $e$ and $e-[e]$ belong to the same class we can always choose $E \subset [0,1]$. This $E$ is called* **Vitali's set**.

**Theorem 1.22** *The outer measure defined for any $A \subset \mathbb{R}$ as*

$$\mu^*(A) = \inf_{A \subset \cup_n I_n} \sum_i L(I_n),$$

*where $I_n \subset \mathbb{R}$ is an open, half-open or a closed interval and $L(I_n)$ is the usual length of the interval, is not additive.*

**Proof** We define a countable set $C = \mathbb{Q} \cap [-1,1]$ (since $C$ is countable we can say that $C = \{c_n\}_{n=1}^{\infty}$), the collection of sets $\{A_n\}_{n=1}^{\infty}$, where $A_n = c_n + E$, and $A = \cup_{n=1}^{\infty} A_n$.
**Claim.**

1. $[0,1] \subset A \subset [-1,2]$;
2. $A_n$ are disjoint sets.

**Proof** Take any $x \in [0,1]$ then there exist unique $e_x \in E$ and $q_x \in \mathbb{Q}$ such that $x = e_x + q_x$. (This is true by definition of the set $E$). But $E \subset [0,1]$ hence $q_x \in [-1,1]$ and therefore any $[0,1] \ni x \in q_x + E$ for some $q_x \in C$. From this it follows $[0,1] \subset A$. Obviously $A \subset [-1,2]$.

Let $A_i = c_i + E$ and $A_j = c_j + E$ and $i \neq j$ ($c_i \neq c_j$). Let's argue by contradiction, assume $A_i \cap A_j \neq \varnothing$ then there exists $x$ such that $x \in c_i + E$ and $x \in c_j + E$, or $x = c_i + y_1 = c_j + y_2$, where $y_1, y_2 \in E$. But then $y_1 - y_2 = c_j - c_i \in \mathbb{Q}$ and this means $y_1 \sim y_2$. Since $E$ contains **exactly one** representative from each class it follows that $y_1 = y_2$ which implies $c_i = c_j$. We got a contradiction, hence $A_i \cap A_j = \varnothing$.

Since $\mu^*$ is translation invariant (prove it!) we have $\mu^*(E) = \mu^*(A_n)$ for all $n$. Suppose $\mu^*$ is additive then by semiadditivity and additivity of $\mu^*$ we have $\sigma$-additivity of $\mu^*$. This means

$$\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n) = \begin{cases} \infty & \text{if } \mu(E) > 0, \\ 0 & \text{if } \mu(E) = 0. \end{cases}$$

However $[0,1] \subset A \subset [-1,2]$ and therefore $1 \leq \mu^*(A) \leq 3$. This is a contradiction. Therefore $\mu^*$ is not additive.

The solution to this problem is to restrict $\mu^*$ to a "nice" collection of subsets where it is additive (and therefore $\sigma$-additive). We call such subsets *measurable.*

There are several equivalent definitions of a measurable set.

**Definition 1.23** *A subset $A \subset X$ is called measurable if*

$$\text{for all } E \subset X, \quad \mu^*(E) = \mu^*(A \cap E) + \mu^*(A^c \cap E)\,. \tag{4}$$

**Definition 1.24** *A subset $A \subset X$ is called measurable if*

$$\mu^*(A \cap X) + \mu^*(A^c \cap X) = 1\,. \tag{5}$$

**Definition 1.25** *A set $A \subset X$ is called measurable if for any $\epsilon > 0$ there exists a set $B \in \mathfrak{R}(\mathfrak{S}_m)$ such that*

$$\mu^*(A \Delta B) < \epsilon$$

**Theorem 1.26** *Definitions 1.23, 1.24, 1.25 are equivalent.*

**Proof Def 1.25 $\Rightarrow$ Def 1.23.** Let $A$ be measurable according to the definition 1.25. Take any $\epsilon > 0$ then there exists a set $B \in \mathfrak{R}(\mathfrak{S}_m)$ such that $\mu^*(A\Delta B) < \epsilon$. Since

$$A \Delta B = A^c \Delta B^c$$

we have $\mu^*(A^c \Delta B^c) < \epsilon$, where $B^c \in \mathfrak{R}(\mathfrak{S}_m)$. Using lemma 1.6, for any $E \subset X$ we obtain

$$|\mu^*(E \cap A) - \mu^*(E \cap B)| \leq \mu^*((E \cap A)\Delta(E \cap B)) \leq \mu^*(A\Delta B) < \epsilon,$$

$$|\mu^*(E \cap A^c) - \mu^*(E \cap B^c)| \leq \mu^*((E \cap A^c)\Delta(E \cap B^c)) \leq \mu^*(A^c\Delta B^c) < \epsilon.$$

From the above inequalities we have

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E \cap B) + \mu^*(E \cap B^c) + 2\epsilon. \tag{6}$$

However since $B \in \mathfrak{R}(\mathfrak{S}_m)$ the following is true:

$$\mu^*(E \cap B) + \mu^*(E \cap B^c) = \mu^*(E).$$

Let's show this: by definition of $\mu^*$ and since $B \in \mathfrak{R}(\mathfrak{S}_m)$ we have

$$\mu^*(E) = \inf_{E \subset \cup_k B_k} \sum_k m'(B_k) = \inf_{E \subset \cup_k B_k} \sum_k (m'(B_k \cap B) + m'(B_k \cap B^c)),$$

where $B_k \in \mathfrak{R}(\mathfrak{S}_m)$. We also have

$$\mu^*(E \cap B) \leq \inf_{E \subset \cup B_k} \sum_k m'(B_k \cap B)$$

and

$$\mu^*(E \cap B^c) \leq \inf_{E \subset \cup B_k} \sum_k m'(B_k \cap B^c).$$

Using the fact $\inf(a+b) \geq \inf a + \inf b$ we obtain

$$\mu^*(E) \geq \mu^*(E \cap B) + \mu^*(E \cap B^c).$$

Applying semiadditivity of $\mu^*$ we have

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c).$$

Now using (6) and taking $\epsilon \to 0$ we get

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \mu^*(E).$$

Now the result follows from semiadditivity of $\mu^*$.
**Def 1.23 $\Rightarrow$ Def 1.24.** This one is obvious.
**Def 1.24 $\Rightarrow$ Def 1.25.** Let $A$ be measurable according to the definition 1.24, i.e.

$$\mu^*(A) + \mu^*(X \backslash A) = 1.$$

For any $\epsilon > 0$ there exist sets $\{B_n\} \subset \mathfrak{R}(\mathfrak{S}_m)$ and $\{C_n\} \subset \mathfrak{R}(\mathfrak{S}_m)$ such that

$$A \subset \cup_n B_n, \quad X \backslash A \subset \cup_n C_n$$

and

$$\sum_n m'(B_n) \leq \mu^*(A) + \epsilon, \quad \sum_n m'(C_n) \leq \mu^*(X \backslash A) + \epsilon.$$

Since $\sum_n m'(B_n) < \infty$ (as $\mu^*(A) \leq 1$) there is $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} m'(B_n) < \epsilon.$$

We define $B = \cup_{n=1}^{N} B_n \in \mathfrak{R}(\mathfrak{S}_m)$ and want to show that $\mu^*(A \Delta B) < 3\epsilon$. It is easy to check that

$$A \Delta B \subset P \cup Q,$$

where $P = \cup_{n=N+1}^{\infty} B_n$ and $Q = \cup_n (B \cap C_n)$. Obviously

$$\mu^*(P) \leq \sum_{n=N+1}^{\infty} m'(B_n) < \epsilon.$$

Let us estimate $\mu^*(Q)$. It is easy to see that $(\cup_n B_n) \cup (\cup_n (C_n \backslash B)) = X$ and hence

$$1 \leq \sum_n m'(B_n) + \sum_n m'(C_n \backslash B).$$

By definition of $B_n$ and $C_n$ we have

$$\sum_n m'(B_n) + \sum_n m'(C_n) \leq \mu^*(A) + \mu^*(X \backslash A) + 2\epsilon = 1 + 2\epsilon$$

and therefore

$$\sum_n m'(C_n \cap B) = \sum_n m'(C_n) - \sum_n m'(C_n \backslash B) < 2\epsilon.$$

This implies $\mu^*(Q) < 2\epsilon$ and $\mu^*(A \Delta B) \leq \mu^*(P) + \mu^*(Q) < 3\epsilon$. This proves the result.

**Remark** Not all sets are measurable. Vitali set, which is used to construct a sequence of subsets of $\mathbb{R}$ on which $\mu^*$ is not $\sigma$-additive, is an example of a *nonmeasurable* set.

**Definition 1.27** *The set function $\mu$ is defined on the collection of all measurable sets $\mathfrak{M}$ by*

$$\mu(A) = \mu^*(A)$$

*for all $A \in \mathfrak{M}$.*

Note that we don't know yet that $\mu$ is a measure.

Let us investigate the properties of measurable sets and $\mu$.

**Theorem 1.28** *The collection $\mathfrak{M}$ of all measurable sets is an algebra.*

**Proof** Let $A_1$ and $A_2$ be measurable sets then for any $\epsilon > 0$ there exist $B_1, B_2 \in \mathfrak{R}(\mathfrak{S}_m)$ such that

$$\mu^*(A_1 \Delta B_1) < \frac{\epsilon}{2}, \quad \mu^*(A_2 \Delta B_2) < \frac{\epsilon}{2}.$$

Using the relation

$$(A_1 \cup A_2)\Delta(B_1 \cup B_2) \subset (A_1 \Delta B_1) \cup (A_2 \Delta B_2)$$

and the fact that $B_1 \cup B_2 \in \mathfrak{R}(\mathfrak{S}_m)$ we obtain

$$\mu^*((A_1 \cup A_2)\Delta(B_1 \cup B_2)) \leq \mu^*(A_1 \Delta B_1) + \mu^*(A_2 \Delta B_2) < \epsilon,$$

therefore $A_1 \cup A_2$ is measurable.

Using the relation

$$(A_1 \backslash A_2)\Delta(B_1 \backslash B_2) \subset (A_1 \Delta B_1) \cup (A_2 \Delta B_2)$$

and the fact that $B_1 \backslash B_2 \in \mathfrak{R}(\mathfrak{S}_m)$ we obtain $A_1 \backslash A_2$ is measurable. The theorem is proved.

**Theorem 1.29** *The function $\mu(A)$ is $\sigma$-additive on the collection $\mathfrak{M}$ of measurable sets*

**Proof** First we show additivity of $\mu$ on $\mathfrak{M}$. Let $A_1, A_2 \in \mathfrak{M}$ and $A_1 \cap A_2 = \varnothing$. For any $\epsilon > 0$ there exist $B_1, B_2 \in \mathfrak{R}(\mathfrak{S}_m)$ such that

$$\mu^*(A_1 \Delta B_1) < \frac{\epsilon}{2}, \quad \mu^*(A_2 \Delta B_2) < \frac{\epsilon}{2}.$$

Define $A = A_1 \cup A_2 \in \mathfrak{M}$ and $B = B_1 \cup B_2$. It is easy to show that

$$B_1 \cap B_2 \subset (A_1 \Delta B_1) \cup (A_2 \Delta B_2)$$

and therefore $m'(B_1 \cap B_2) < \epsilon$. By lemma 1.6 we have

$$|m'(B_1) - \mu^*(A_1)| < \frac{\epsilon}{2}, \quad |m'(B_2) - \mu^*(A_2)| < \frac{\epsilon}{2}.$$

Since $m'$ is additive on $\mathfrak{R}(\mathfrak{S}_m)$ we obtain

$$m'(B) = m'(B_1) + m'(B_2) - m'(B_1 \cap B_2) \geq \mu^*(A_1) + \mu^*(A_2) - 2\epsilon.$$

Noting that $A \Delta B \subset (A_1 \Delta B_1) \cup (A_2 \Delta B_2)$ and using semiadditivity of $\mu^*$ we have

$$\mu^*(A) \geq m'(B) - \mu^*(A \Delta B) \geq m'(B) - \epsilon \geq \mu^*(A_1) + \mu^*(A_2) - 3\epsilon.$$

Since $\epsilon >$ is arbitrary we have

$$\mu^*(A) \geq \mu^*(A_1) + \mu^*(A_2).$$

Using semiadditivity of $\mu^*$ and the fact that $A_1, A_2, A \in \mathfrak{M}$ we obtain

$$\mu(A) = \mu(A_1) + \mu(A_2).$$

and hence $\mu$ is additive.

Using theorem 1.8 and the fact that $\mu$ is semiadditive on $\mathfrak{M}$ (since on $\mathfrak{M}$ it coincides with $\mu^*$ and $\mu^*$ is semiadditive) we obtain the result.

Now we know that $\mu$ is a $\sigma$-additive measure.

**Theorem 1.30** *The collection $\mathfrak{M}$ of all measurable sets is a $\sigma$-algebra.*

**Proof** Let $A_i \in \mathfrak{M}$ for $i = 1, 2, ...$ and $A = \cup_{i=1}^{\infty} A_i$. Define

$$A'_n = A_n \setminus \cup_{i=1}^{n-1} A_i.$$

It is clear that $A'_n$ are measurable (by theorem 1.28), disjoint and $A = \cup_{n=1}^{\infty} A'_n$. By theorem 1.29 we have: for all $N \in \mathbb{N}$

$$\sum_{n=1}^{N} \mu(A'_n) = \mu(\cup_{n=1}^{N} A'_n) \leq \mu(A).$$

Therefore the series $\sum_{n=1}^{\infty} \mu(A'_n)$ converges and for any $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $\sum_{n=M}^{\infty} \mu(A'_n) < \frac{\epsilon}{2}$. The set $C = \cup_{n=1}^{M} A'_n \in \mathfrak{M}$ and hence there exist $B \in \mathfrak{R}(\mathfrak{S}_m)$ such that $\mu^*(C \Delta B) < \frac{\epsilon}{2}$. Since

$$A \Delta B \subset (C \Delta B) \cup (\cup_{n=M}^{\infty} A'_n)$$

we obtain

$$\mu^*(A \Delta B) < \epsilon$$

and hence $A$ is measurable. Since $\mathfrak{M}$ is an algebra the theorem is proved.

**Theorem 1.31** *Measure $\mu$ is complete.*

**Proof** Let $A \in \mathfrak{M}$, $B \subset A$ and $\mu(A) = 0$, then $\mu^*(B \Delta \varnothing) \leq \mu^*(A \Delta \varnothing) = \mu(A) = 0$. Since $\varnothing \in \mathfrak{R}(\mathfrak{S}_m)$ we obtain $B \in \mathfrak{M}$. The theorem is proved.

We showed that the extension $\mu$ of a measure $m$ from a semi-algebra $\mathfrak{S}_m$ to the $\sigma$-algebra $\mathfrak{M} \supset \mathfrak{S}_m$ of all measurable sets coinciding on $\mathfrak{M}$ with the outer measure $\mu^*$ is complete $\sigma$-additive measure. It seems that we constructed one complete measure $\bar{\mu}$ in section 1.1, theorem 1.16 and another measure $\mu = \mu^* \restriction_{\mathfrak{M}}$ here. In fact these two measures coincide.

**Theorem 1.32** *If $m'$ is a $\sigma$-additive $\sigma$-finite measure on a ring $\mathfrak{R}$ and if $\mu^*$ is the outer measure induced by $m'$ then the completion of the extension of $m'$ to the $\sigma$-algebra $\mathfrak{P}(\mathfrak{R})$ is identical with restriction of $\mu^*$ to the class of all $\mu^*$ measurable sets.*

**Proof** The proof of this theorem is left as an exercise.

## Problems

1. Let $A_1, A_2, \ldots$ be an increasing sequence of subsets of $X$, i. e., $A_j \subset A_{j+1}\ \forall j \in \mathbb{N}$. Suppose that $A_j$ is $\mu$-measurable for all $j \in \mathbb{N}$ and prove that $\mu(\bigcup_{j=1}^{\infty} A_j) = \lim_{j\to\infty} \mu(A_j)$.

2. Let $A$ be a Lebesgue measurable subset of $\mathbb{R}$. Prove that, for each $\varepsilon > 0$, there exists an open subset $E_\varepsilon$ of $\mathbb{R}$ such that
$$A \subset E_\varepsilon \quad \text{and} \quad \mu(E_\varepsilon \setminus A) < \varepsilon.$$

3. A subset of $\mathbb{R}^n$ is called a *rectangle* if it is a product of intervals, i.e. $\mathcal{R} \subset \mathbb{R}^n$ is a rectangle if there exist intervals $I_1, I_2, \ldots, I_n \subset \mathbb{R}$ such that $\mathcal{R} = I_1 \times I_2 \times \ldots \times I_n$. Prove that every open subset of $\mathbb{R}^n$ can be written as a countable union of open rectangles. Deduce that the open subsets and closed subsets of $\mathbb{R}$ are all measurable.

4. i. Let $A$ be a Lebesgue measurable subset of $\mathbb{R}$. Prove that, given $\varepsilon > 0$, there exists a closed subset $F_\varepsilon$ of $\mathbb{R}$ such that
$$F_\varepsilon \subset A \quad \text{and} \quad \mu(A \setminus F_\varepsilon) < \varepsilon.$$

   ii. Let $B$ be a subset of $\mathbb{R}$ with the property that, for each $\varepsilon > 0$, there exists an open subset $E_\varepsilon$ of $\mathbb{R}$ such that
$$B \subset E_\varepsilon \quad \text{and} \quad \mu^*(E_\varepsilon \setminus B) < \varepsilon.$$

   Prove that $B$ is Lebesgue measurable.

5. Given a sequence of subsets $E_1, E_2, \ldots$ of a set $X$ we define
$$\limsup E_j := \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} E_k \qquad \text{and} \qquad \liminf E_j := \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} E_k\,.$$

Note that $\limsup E_j$ is the set of points which belong to $E_j$ for infinitely many values of $j$. Suppose that $E_j$ is $\mu$-measurable for all $j \in \mathbb{N}$ and prove that

$$\mu(\liminf E_j) \leqslant \liminf \mu(E_j)\,.$$

6. Let $U$ be an open subset of $\mathbb{R}$. For each $x \in U$, let

   $$a_x := \inf\{a \in \mathbb{R} \mid (a, x) \subset U\}, b_x := \sup\{b \in \mathbb{R} \mid (x, b) \subset U\}, I_x := (a_x, b_x)\,.$$

   Prove that $x \in U \Rightarrow I_x \subset U$ and that, if $x, y \in U$ and $I_x \cap I_y \neq \emptyset$ then $I_x = I_y$. Deduce that **every** open subset of $\mathbb{R}$ is a countable union of disjoint open intervals.

7. Given $\lambda \in \mathbb{R}$ and $A \subset \mathbb{R}$, let $A_{+\lambda} := \{x + \lambda \mid x \in A\}$.

   i. Prove that, for all $A \subset \mathbb{R}$ and for all $\lambda \in \mathbb{R}$, $\mu^*(A_{+\lambda}) = \mu^*(A)$ ($\mu^*$ is the outer measure from the theorem 1.22).

   ii. Prove that if $A \subset \mathbb{R}$ is Lebesgue measurable and $\lambda \in \mathbb{R}$ then $A_{+\lambda}$ is Lebesgue measurable.

## 2 Measurable functions

First we give some general definitions.

**Definition 2.1** *$(X, \mathfrak{M})$ is called a measurable space if $X$ is some set, $\mathfrak{M}$ is a $\sigma$-algebra on $X$.*

**Definition 2.2** *The triple $(X, \mathfrak{M}, \mu)$ is called a measure space if $X$ is some set, $\mathfrak{M}$ is a $\sigma$-algebra of subsets of $X$ and $\mu$ is a on $\mathfrak{M}$.*

**Definition 2.3** *A function $f : X \to \mathbb{R}$ is called $\mu$-measurable (or just measurable) if*

$$f^{-1}(A) \in \mathfrak{M}$$

*for any Borel set $A$ on $\mathbb{R}$.*

Our main interest in measurable functions lies in the theory of **Lebesgue integration**. Therefore throughout the rest of the lecture notes $X \subset \mathbb{R}^n$, $\mathfrak{M}$ is Borel algebra with all null sets and $\mu$ is Lebesgue measure, although the theory remains true for general measure spaces.

**Proposition 2.4** *Function $f : X \to \mathbb{R}$ is measurable if and only if for any $c \in \mathbb{R}$ set*

$$\{x \in X \;:\; f(x) < c\}$$

*is measurable.*

**Proof** Necessity is obvious since $(-\infty, c)$ is Borel set and hence measurable. **Sufficiency:** It is not difficult to show that $\sigma$-algebra created by sets $(-\infty, c)$, where $c \in \mathbb{R}$, coincides with Borel $\sigma$-algebra on $\mathbb{R}$. If $\{x \in X \;:\; f(x) < c\}$ is measurable for all $c \in \mathbb{R}$ then $f^{-1}(-\infty, c) \in \mathfrak{M}$ (by definition of inverse image). From this it follows that $\mathfrak{P}(f^{-1}(-\infty, c)) \in \mathfrak{M}$ and therefore $f^{-1}(\mathfrak{P}((-\infty, c))) \in \mathfrak{M}$.

**Exercise 2.1** *In the theorem we have used the fact that if $\mathfrak{A}$ is a collection of sets then $\mathfrak{P}(f^{-1}(\mathfrak{A})) = f^{-1}(\mathfrak{P}(\mathfrak{A}))$. Prove it.*

**Proposition 2.5** *Let $f : X \to \mathbb{R}$ be some function. The following statements are equivalent*

1. *$\{x \in X \;:\; f(x) < c\} \in \mathfrak{M}$ for any $c \in \mathbb{R}$;*
2. *$\{x \in X \;:\; f(x) \geq c\} \in \mathfrak{M}$ for any $c \in \mathbb{R}$;*

*3.* $\{x \in X\ :\ f(x) > c\} \in \mathfrak{M}$ *for any* $c \in \mathbb{R}$*;*

*4.* $\{x \in X\ :\ f(x) \leq c\} \in \mathfrak{M}$ *for any* $c \in \mathbb{R}$*;*

**Proof** Since $\mathfrak{M}$ is a $\sigma$-algebra it is easy to see that statements 1 and 2 are equivalent and statements 3 and 4 are equivalent. Using the facts that

$$\{x \in X\ :\ f(x) \geq c\} = \cap_{n=1}^{\infty}\{x \in X\ :\ f(x) > a - \frac{1}{n}\}$$

and

$$\{x \in X\ :\ f(x) < c\} = \cup_{n=1}^{\infty}\{x \in X\ :\ f(x) \leq a - \frac{1}{n}\}$$

We have the result.

Now to find out if the function is measurable we just have to check either of points $1-4$.

We also want to know what kind of operation we may do with measurable functions that the resulting function is also measurable. For instance, we want to know if sum, product, e.t.c of measurable functions is measurable.

**Lemma 2.1** *Let* $f : X \to \mathbb{R}$ *be* $\mu$*-measurable and* $\phi : \mathbb{R} \to \mathbb{R}$ *be Borel measurable. Then* $\phi(f(x))$ *is* $\mu$*-measurable.*

**Proof** Let $g(x) = \phi(f(x))$ and $A \subset \mathbb{R}$ be an arbitrary Borel set. Then $\phi^{-1}(A)$ is Borel set since $\phi$ is Borel measurable and $g^{-1}(A) = f^{-1}(\phi^{-1}(A))$ is $\mu$-measurable. Lemma is proved.

**Theorem 2.6** *Let* $f : X \to \mathbb{R}$ *and* $g : X \to \mathbb{R}$ *be measurable functions. Then* $f+g$*,* $f-g$*,* $fg$*,* $\frac{f}{g}$ *(if* $g(x) \neq 0$*),* $\max(f,g)$ *and* $\min(f,g)$ *are measurable functions.*

**Proof** It is obvious that if $f$ is measurable function then so are $cf$ and $f(x)+c$ for any $c \in \mathbb{R}$. If $f$ and $g$ are measurable functions we show that set $\{x \in X\ :\ f(x) > g(x)\}$ is measurable. Indeed, take $\{r_k\}_{k=1}^{\infty}$ - the sequence of **all** rational numbers (we can do it since rational numbers are countable). Then

$$\{x \in X : f(x) > g(x)\} = \cup_{k=1}^{\infty}(\{x \in X : f(x) > r_k\} \cap \{x \in X : r_k > g(x)\}).$$

Therefore we have that set $\{x \in X\ :\ f(x) > -g(x) + c\}$ is measurable. Hence we obtain $f+g$ is a measurable function.

To show that $fg$ is measurable we use the following identity

$$fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$$

Using $f+g$, $f-g$ are measurable functions and the fact that continuous function of a measurable function is itself measurable we conclude the proof.

The rest of the proof is left as an exercise.

**Theorem 2.7** *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions. Then $\sup_n f_n(x)$, $\inf_n f_n(x)$, $\limsup_n f_n(x)$ and $\liminf_n f_n(x)$ are measurable functions.*

**Proof** Let $g(x) = \sup_n f_n(x)$ then for any $c \in \mathbb{R}$ we have

$$\{x \in X \,:\, g(x) > c\} = \cup_n \{x \in X \,:\, f_n(x) > c\}$$

and hence $g(x)$ is a measurable function.

Let $g(x) = \inf_n f_n(x)$ then for any $c \in \mathbb{R}$ we have

$$\{x \in X \,:\, g(x) < c\} = \cup_n \{x \in X \,:\, f_n(x) < c\}$$

and hence $g(x)$ is a measurable function.

By definition we have

$$\limsup_n f_n(x) = \inf_k \sup_{n \geq k} f_n(x)$$

and

$$\liminf_n f_n(x) = \sup_k \inf_{n \geq k} f_n(x)$$

hence result follows from previous arguments..

**Exercise 2.2** *From this theorem it is easy to deduce that if $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions that converges pointwise to a function $f(x)$ then $f(x)$ is a measurable function. Do it.*

We did not use anything about completeness of our measure yet. Now is the time.

**Definition 2.8** *Functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are equivalent ($f \sim g$) if*

$$\mu(\{x \in X \,:\, g(x) \neq f(x)\}) = 0.$$

**Proposition 2.9** *A function $f : X \to \mathbb{R}$ equivalent to some measurable function $g : X \to \mathbb{R}$ is measurable itself.*

**Proof** By definition of equivalence sets $\{x \in X \,:\, f(x) \leq c\}$ and $\{x \in X \,:\, g(x) \leq c\}$ may differ just by some null set and hence if one is measurable the other is measurable as well.

## 2.1 Convergence of measurable functions

In this section we define some types of convergences of function sequences on the space $(X, \mu)$.

**Definition 2.10** *A sequence of measurable functions $\{f_n(x)\}_{n=1}^{\infty}$, defined on $(X, \mu)$ is called convergent almost everywhere to $f(x)$ ($f_n(x) \to f(x)$ a.e. $X$) if*

$$\mu(\{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\}) = 0$$

**Definition 2.11** *A sequence of measurable functions $\{f_n(x)\}_{n=1}^{\infty}$, defined on $(X, \mu)$ is called convergent in measure to $f(x)$ ($f_n(x) \to^{\mu} f(x)$ a.e. $X$) if for every $\delta > 0$*

$$\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| \geq \delta\}) = 0$$

**Proposition 2.12** *If a sequence of measurable functions $\{f_n(x)\}_{n=1}^{\infty}$ converges almost everywhere to a function $f(x)$ then $f(x)$ is also a measurable function*

**Proof** The proof is left as an exercise.

Let us first prove the theorem that relates the notion of convergence a.e. and uniform convergence.

**Theorem 2.13** *(Egoroff) Suppose that a sequence of measurable functions $\{f_n(x)\}_{n=1}^{\infty}$ converges a.e to $f(x)$ on $X$ ($\mu(X) < \infty$). Then for every $\delta > 0$ there exists a measurable set $X_\delta \subset X$ such that*

1. *$\mu(X_\delta) > \mu(X) - \delta$;*
2. *the sequence $f_n(x)$ converges to $f(x)$ uniformly on $X_\delta$.*

**Proof** Obviously $f(x)$ is measurable. We define

$$X_n^m = \cap_{i\geq n}\{x \in X : |f_i(x) - f(x)| < \frac{1}{m}\}$$

and $X^m = \cup_{n=1}^{\infty} X_n^m$. By definition of $X_n^m$ we see that $X_1^m \subset X_2^m \subset ... \subset X_n^m \subset ....$ By continuity of measure we have: for any $m$ and any $\delta > 0$ there exists $n_0(m)$ such that

$$\mu(X^m \backslash X_{n_0(m)}^m) < \frac{\delta}{2^m}.$$

We define $X_\delta = \cap_{m=1}^{\infty} X_{n_0(m)}^m$. Let us show that $X_\delta$ is the required set.

1. $f_n \to f$ uniformly on $X_\delta$ since if $x \in X_\delta$ then $x \in X^m_{n_0(m)}$ for any $m$ and hence $|f_i(x) - f(x)| < \frac{1}{m}$ if $i \geq n_0(m)$. This is exactly the definition of uniform convergence.

2. Let us estimate $\mu(X \backslash X_\delta)$. We notice that $\mu(X \backslash X^m) = 0$ for any $m$. Indeed, if $x_0 \in X \backslash X^m$ then there exists a sequence $i \to \infty$ such that $|f_i(x_0) - f(x_0)| \geq \frac{1}{m}$. This means that $f_i(x_0)$ does not converge to $f(x_0)$. Since $f_i(x) \to f(x)$ a.e. $X$ we have $\mu(X \backslash X^m) = 0$.

   This implies $\mu(X \backslash X^m_{n_0(m)}) = \mu(X^m \backslash X^m_{n_0(m)}) < \frac{\delta}{2^m}$ and we obtain

$$\begin{aligned}\mu(X \backslash X_\delta) =& \mu(X \backslash \cap_{m=1}^\infty X^m_{n_0(m)}) \\ =& \mu(\cup_{m=1}^\infty (X \backslash X^m_{n_0(m)})) \leq \sum_{m=1}^\infty \mu(X \backslash X^m_{n_0(m)}) \leq \delta.\end{aligned}$$

The theorem is proved.

In the two theorems below we relate convergence a.e. and convergence in measure.

**Theorem 2.14** *If the sequence of measurable functions $f_n(x) \to f(x)$ a.e. then $f_n(x) \to^\mu f(x)$.*

**Proof** It is easy to see that $f(x)$ is measurable. Let $A = \{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\}$, obviously $\mu(A) = 0$. Fix $\delta > 0$ and define $X_k(\delta) = \{x \in X : |f_k(x) - f(x)| \geq \delta\}$, $R_n(\delta) = \cup_{k \geq n} X_k(\delta)$, and $M = \cap_{n=1}^\infty R_n(\delta)$. Obviously $R_1(\delta) \supset R_2(\delta) \supset ....$ By continuity of the measure we have $\mu(R_n(\delta)) \to \mu(M)$ as $n \to \infty$.

Let us show that $M \subset A$. Take $x_0 \notin A$, for this point we have: for any $\delta > 0$ there exists $N$ such that $|f_k(x_0) - f(x_0)| < \delta$ for any $k \geq N$. Therefore $x_0 \notin R_N(\delta)$ and hence $x_0 \notin M$. This implies $\mu(R_n(\delta)) \to 0$ and since $X_n(\delta) \subset R_n(\delta)$ we obtain $\mu(X_n(\delta)) \to 0$. The theorem is proved.

**Theorem 2.15** *If a sequence of measurable functions $f_n \to^\mu f$ then there exists a subsequence $\{f_{n_k}\} \subset \{f_n\}$ that converges to $f$ a.e. $X$.*

**Proof** Let $\{\epsilon_n\}$ be a positive sequence such that $\epsilon_n \to 0$ and let $\{\eta_n\}$ be a positive sequence such that $\sum_{n=1}^\infty \eta_n < \infty$. Let us build a sequence of indices $n_1 < n_2 < ...$ as follows:
choose $n_1$ to be such that $\mu\{x \in X : |f_{n_1}(x) - f(x)| \geq \epsilon_1\} < \eta_1$;
choose $n_2 > n_1$ to be such that $\mu\{x \in X : |f_{n_1}(x) - f(x)| \geq \epsilon_2\} < \eta_2$ e.t.c.

We show that $f_{n_k}(x) \to f(x)$ a.e. $X$. Indeed, let $R_i = \cup_{k=i}^{\infty}\{x \in X : |f_{n_k}(x) - f(x)| \geq \epsilon_k\}$, $M = \cap_{i=1}^{\infty} R_i$. Obviously $R_1 \supset R_2 \supset ...$, using the continuity of the measure we obtain $\mu(R_i) \to \mu(M)$, but $\mu(R_i) \leq \sum_{k=i}^{\infty} \eta_k$ hence $\mu(R_i) \to 0$, since the series converges.

Now we have to check that $f_{n_k}(x) \to f(x)$ in $X \backslash M$. Let $x_0 \in X \backslash M$ then there exists $i_0$ such that $x_0 \notin R_{i_0}$ and hence for any $k \geq i_0$ $x_0 \notin .\{x \in X : |f_{n_k}(x) - f(x)| \geq \epsilon_k\}$. But this implies $|f_{n_k}(x) - f(x)| < \epsilon_k$ for any $k \geq i_0$. Since $\epsilon_k \to 0$ we get that $f_{n_k}(x_0) \to f(x_0)$. The theorem is proved.

**Theorem 2.16** *(Lusin) A function $f : [a,b] \to \mathbb{R}$ is measurable if and only if for any $\epsilon > 0$ there exists a continuous function $\phi_\epsilon$ such that*

$$\mu\{x \in [a,b] \,:\, f(x) \neq \phi_\epsilon(x)\} < \epsilon$$

**Proof** Let for any $\epsilon > 0$ there exists $\phi_\epsilon$ - continuous function such that

$$\mu\{x \in [a,b] \,:\, f(x) \neq \phi_\epsilon(x)\} < \epsilon$$

It is easy to see that if $A = \{x \in [a,b] \,:\, f(x) < c\}$ and $B = \{x \in [a,b] : \phi_\epsilon(x) < c\}$ then

$$A \subset B \cup \{x \in [a,b] \,:\, f(x) \neq \phi_\epsilon(x)\},$$

$$B \subset A \cup \{x \in [a,b] \,:\, f(x) \neq \phi_\epsilon(x)\}.$$

Therefore $A \Delta B \subset \{x \in [a,b] \,:\, f(x) \neq \phi_\epsilon(x)\}$ and $\mu^*(A\Delta B) < epsion$. On the other hand since $\phi_\epsilon$ is continuous then it is measurable and set $B$ is measurable. Therefore there exists Borel set $C$ such that $\mu^*(B\Delta C) < \epsilon$ (actually equal to 0). From this it follows that $\mu^*(A\Delta C) < 2\epsilon$ and hence $A$ is measurable.

The second part of the proof may be done using Egoroff theorem. It is left as an exercise.

# 3 Lebesgue integral

We are going to define Lebesgue integral for elementary functions first.

**Definition 3.1** *A function $f : X \to \mathbb{R}$ is called elementary if it is measurable and takes not more than a countable number of values.*

**Proposition 3.2** *A function $f : X \to \mathbb{R}$ taking not more than a countable number of values $y_1, y_2, \ldots$ is measurable if and only if all sets $A_n = \{x \in X : f(x) = y_n\}$ are measurable.*

**Proof** The necessity follows from the fact that $A_n = f^{-1}(y_n)$ and $\{y_n\}$ are Borel sets. The sufficiency is clear since for any $A \in \mathfrak{P}(\mathbb{R})$ we have $f^{-1}(A) = \cup_{y_n \in A} A_n$, where the union is at most countable. Hence $f^{-1}(A) \in \mathfrak{M}$.

**Proposition 3.3** *A function $f : X \to \mathbb{R}$ is measurable if and only if it is a limit of a* **uniformly** *convergent sequence of elementary functions.*

**Proof** Let $\{f_n(x)\}$ be a sequence of elementary functions and $f_n \to f$ uniformly on $X$. Then obviously $f_n(x) \to f(x)$ a.e. and hence $f(x)$ is measurable by theorem 2.12. Now let $f(x)$ be a measurable function. We set $f_n(x) = \frac{m}{n}$ on $A_m^n = \{x \in X : \frac{m}{n} \leq f(x) < \frac{m+1}{n}\}$ ($m \in \mathbb{Z}$ and $n \in \mathbb{N}$). Obviously $f_n(x)$ is elementary and $|f_n(x) - f(x)| \leq \frac{1}{n}$ on $X$. Taking $n \to \infty$ we get the result.

Let us define Lebesgue integral for elementary functions. Take $f : X \to \mathbb{R}$ be elementary function with values

$$y_1, y_2, \ldots, y_n, \ldots \qquad (y_i \neq y_j \text{ for } i \neq j).$$

Let $A \subset X$ be a measurable set. We define

$$\int_A f(x) d\mu = \sum_n y_n \mu(A_n), \tag{7}$$

where $A_n = \{x \in A : f(x) = y_n\}$.

**Definition 3.4** *An elementary function $f : X \to \mathbb{R}$ is integrable on $A$ if the series (7) is* **absolutely** *convergent. In this case (7) is called an integral of $f$ over $A$.*

**Lemma 3.1** *Let $A = \cup_k B_k$, $B_i \cap B_j = \varnothing$ for $i \neq j$ and on any $B_k$ function $f : A \to \mathbb{R}$ takes only one value $c_k$. Then*

$$\int_A f(x)d\mu = \sum_k c_k \mu(B_k) \tag{8}$$

*and $f$ is integrable on $A$ if and only if the series in (8) converges absolutely.*

**Proof** Obviously $f(x)$ is elementary and then we can find at most countable number of distinct values $y_1, y_2, ..., y_n, ...$ of $f(x)$. We have $A_n = \{x \in A : f(x) = y_n\} = \cup_{c_k = y_n} B_k$ and therefore

$$\sum_n y_n \mu(A_n) = \sum_n y_n \sum_{c_k = y_n} \mu(B_k) = \sum_k c_k \mu(B_k)$$

Now we have to show that these two series converge or diverge simultaneously and this is true since

$$\sum_n |y_n| \mu(A_n) = \sum_n |y_n| \sum_{c_k = y_n} \mu(B_k) = \sum_k |c_k| \mu(B_k).$$

The lemma is proved.

It is easy to see that integral of elementary function is linear functional. Now we want to extend the definition of Lebesgue integral to measurable functions that are not necessarily elementary.

## 3.1 Integrable functions

**Definition 3.5** *A function $f : X \to \mathbb{R}$ is integrable on a measurable set $A \subset X$ if there exists a sequence $\{f_n(x)\}$ of elementary* **integrable** *functions on $A$ such that $f_n \to f$* **uniformly** *on $A$. The limit*

$$I = \lim_{n \to \infty} \int_A f_n(x) d\mu \tag{9}$$

*is denoted by*

$$\int_A f(x) d\mu$$

*and is called the integral of $f$ over $A$.*

This definition makes sense if the following conditions hold:

1. The limit (9) exists for any uniformly convergent sequence of elementary integrable functions.
2. For a fixed $f(x)$ this limit is independent of the choice of the sequence $\{f_n(x)\}$.
3. If $f(x)$ is an elementary function then this definition of integrability coincides with the definition 3.4

Let us show that all these points are satisfied. Notice that if $\{f_n\}$ is a sequence of elementary integrable functions then

$$\left|\int_A f_n(x)d\mu - \int_A f_m(x)d\mu\right| \leq \mu(A) \sup_{x\in A} |f_n(x) - f_m(x)|. \tag{10}$$

This inequality implies that if $\{f_n\}$ converges uniformly to $f$ then $\int_A f_n(x)d\mu$ is a Cauchy sequence and hence $\lim_{n\to\infty} \int_A f_n(x)d\mu$ exists. Point 1 is proved.

To show point 2 we assume that there are two sequences $\{f_n\}$ and $\{g_n\}$ of elementary integrable functions uniformly converging to $f$. Obviously we have $\sup_{x\in A} |f_n(x) - g_n(x)| \to 0$ as $n \to \infty$. Formula (10) implies $\left|\int_A f_n(x)d\mu - \int_A g_n(x)d\mu\right| \to 0$ that proves point2.

To show point 3 take $f_n(x) = f(x)$ for all $n$, where $f$ is elementary and integrable and then use point 2.

### Properties of the integral

Let $f$ and $g$ be any integrable functions on $A$ then:

1. $\int_A 1 d\mu = \mu(A)$;
2. for any $c \in \mathbb{R}$ $\int cf(x)d\mu = c\int_A f(x)d\mu$;
3. $\int_A (f(x) + g(x))d\mu = \int_A f(x)d\mu + \int_A g(x)d\mu$;
4. if $f(x) \geq 0$ then $\int_A f(x)d\mu \geq 0$;
5. if $\mu(A) = 0$ then $\int_A f(x)d\mu = 0$;
6. if $f(x) = g(x)$ a.e. then $\int_A f(x)d\mu = \int_A g(x)d\mu$;
7. any bounded measurable function is integrable;
8. if $h(x)$ is measurable function on $A$ and $|h(x)| \leq |f(x)|$ for some integrable $f$ then $h(x)$ is integrable;

9. for any measurable function $h(x)$ integrals $\int_A h(x)d\mu$ and $\int_A |h(x)|d\mu$ exist or don't exist simultaneously.

These properties are usually proved for the integrals of elementary functions and then, passing to the limit, for integrable functions. We prove here only property 8..

**Proposition 3.6** *If a function $f(x)$ is integrable and measurable function $|h(x)| \leq f(x)$ then $h(x)$ is also integrable.*

**Proof** Let $f(x)$ and $h(x)$ be elementary functions. Then $A$ can be written as a union of countable number of disjoint sets $A_n$ on each of which $f(x)$ and $h(x)$ are constants:

$$h(x) = a_n, \quad f(x) = b_n \text{ and } |a_n| \leq b_n.$$

Obviously

$$\sum_n |a_n|\mu(A_n) \leq \sum_n b_n\mu(A_n) = \int_A f(x)d\mu.$$

This implies $h(x)$ is integrable and

$$\left|\int_A h(x)d\mu\right| \leq \int_A f(x)d\mu.$$

Now let $f(x)$ be an integrable function and $|h(x)| \leq f(x)$. We may approximate $h(x)$ and $f(x)$ by sequences of uniformly convergent elementary functions $\{h_n(x)\}$ and $\{f_n(x)\}$, respectively. Since $f(x)$ is integrable then $f_n(x)$ can be chosen as elementary **integrable** functions. This implies that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then

$$|h_n(x) - h(x)| < \epsilon \text{ and } |f_n(x) - f(x)| < \epsilon.$$

Obviously for $n > N$ $|h_n(x)| \leq |f_n(x)| + 2\epsilon$ and then as before $h_n(x)$ are integrable. Hence $h(x)$ is a limit of uniformly convergent sequence of elementary integrable functions and therefore is integrable. Proposition is proved.

**Proposition 3.7** *Let $A = \cup_n A_n$, where $A_n$ are measurable sets and $A_i \cap A_j = \varnothing$ for $i \neq j$, and let $f : A \to \mathbb{R}$ be an integrable function then*

$$\int_A f(x)d\mu = \sum_n \int_{A_n} f(x)d\mu$$

*and existence of left integral implies existence of integrals in the right and absolute convergence of the series.*

**Proof** We check the theorem for integrable elementary functions first and then pass to the limit to get the proof for any integrable function. Let $f(x)$ be an elementary integrable function taking values $y_1, y_2, \ldots$. Let $B_k = \{x \in A : f(x) = y_k\}$ and $B_k^n = \{x \in A_n : f(x) = y_k\}$ then

$$\begin{aligned}\int_A f(x)d\mu &= \sum_k y_k \mu(B_k) = \sum_k y_k \sum_n \mu(B_k^n) \\ &= \sum_n \sum_k y_k \mu(B_k^n) = \sum_n \int_{A_n} f(x)d\mu.\end{aligned}$$

We can change summation indices since $f$ is an integrable elementary function.

Now let $f$ be any integrable function, by definition 3.5 for every $\epsilon > 0$ we may find an elementary integrable function $g_\epsilon$ such that $|g_\epsilon(x) - f(x)| < \epsilon$ on $A$. For $g_\epsilon$ we have

$$\int_A g_\epsilon(x)d\mu = \sum_n \int_{A_n} g_\epsilon(x)d\mu.$$

Since $g_\epsilon$ is integrable over each $A_n$ we have that $f$ is integrable over each $A_n$ and

$$\sum_n \left| \int_{A_n} f(x)d\mu - \int_{A_n} g(x)d\mu \right| \leq \sum_n \epsilon\mu(A_n) = \epsilon\mu(A),$$

$$\left| \int_A f(x)d\mu - \int_A g(x)d\mu \right| \leq \epsilon\mu(A).$$

Therefore the series $\sum_n \int_{A_n} f(x)d\mu$ converges absolutely and

$$\left| \sum_n \int_{A_n} f(x)d\mu - \int_A f(x)d\mu \right| \leq 2\epsilon\mu(A).$$

Letting $\epsilon \to 0$ we get the result.

**Proposition 3.8** *Let $A = \cup_n A_n$, where $A_n$ are measurable sets and $A_i \cap A_j = \varnothing$ for $i \neq j$. Let $f : A \to \mathbb{R}$ be a measurable function and $\int_{A_n} f(x)d\mu$ exist for all $n$ and the series $\sum_n \int_{A_n} |f(x)|d\mu$ converges. Then*

$$\int_A f(x)d\mu = \sum_n \int_{A_n} f(x)d\mu.$$

**Proof** We check the theorem for elementary functions first and then pass to the limit to get the proof for any integrable function. Let $f(x)$ be an elementary function taking values $y_1, y_2, ....$ Let $B_k = \{x \in A \,:\, f(x) = y_k\}$ and $B_k^n = \{x \in A_n \,:\, f(x) = y_k\}$ then

$$\int_{A_n} |f(x)| d\mu = \sum_k |y_k| \mu(B_k^n).$$

Therefore

$$\sum_n \int_{A_n} |f(x)| d\mu = \sum_n \sum_k |y_k| \mu(B_k^n) = \sum_k |y_k| \mu(B_k).$$

Hence $f$ is integrable over $A$ and $\int_A f(x) d\mu = \sum_k y_k \mu(B_k)$.

Now let $f$ be any measurable function, by proposition 3.3 for every $\epsilon > 0$ we may find an elementary function $g_\epsilon$ such that $|g_\epsilon(x) - f(x)| < \epsilon$ on $A$. For $g_\epsilon$ we have

$$\int_{A_n} |g_\epsilon(x)| d\mu \leq \int_{A_n} f(x) d\mu + \epsilon \mu(A_n).$$

Therefore $\sum_n \int_{A_n} |g_\epsilon(x)| d\mu$ converges and $g_\epsilon(x)$ is integrable. But then $f(x)$ is integrable too and by previous proposition we have the result.

**Theorem 3.9** *(Chebyshev inequality) Let $f(x) \geq 0$ be integrable function on $A$ and $c > 0$ be some positive constant. Then*

$$\mu(\{x \in A \,:\, f(x) \geq c\}) \leq \frac{1}{c} \int_A f(x) d\mu.$$

**Proof** Take $B = \{x \in A \,:\, \phi(x) \geq c\}$ then

$$\int_A \phi(x) d\mu = \int_B \phi(x) d\mu + \int_{A \setminus B} \phi(x) d\mu \geq \int_B \phi(x) d\mu \geq c\mu(B).$$

The theorem is proved.

**Corollary 3.10** *If $\int_A |f(x)| d\mu = 0$ then $f(x) = 0$ a.e. on $A$.*

**Proof** By Chebyshev inequality we have

$$\mu(\{x \in A \,:\, f(x) \geq \frac{1}{n}\}) \leq n \int_A f(x) d\mu \quad \text{for any } n.$$

This implies

$$\mu(\{x \in A : f(x) \neq 0\}) \leq \sum_{n=1}^{\infty} \mu(\{x \in A : f(x) \geq \frac{1}{n}\}) = 0.$$

The corollary is proved.

**Theorem 3.11** *(Absolute continuity of the integral) Let $f(x)$ be an integrable function on $A$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\left| \int_E f(x) d\mu \right| < \epsilon$$

*for all measurable $E \subset A$ such that $\mu(E) < \delta$.*

**Proof** Fix $\epsilon > 0$. The theorem is obvious if $f$ is a bounded function. Let $f$ be an arbitrary integrable function on $A$. We define $A_n = \{x \in A : n \leq |f(x)| < n+1\}$, $B_n = \cup_{k=0}^{n} A_n$ and $C_n = A \backslash B_n$. By proposition 3.7 we have

$$\int_A |f(x)| d\mu = \sum_{n=0}^{\infty} \int_{A_n} |f(x)| d\mu.$$

Choose $N$ such that

$$\int_{C_N} |f(x)| d\mu = \sum_{n=N+1}^{\infty} \int_{A_n} |f(x)| d\mu < \frac{\epsilon}{2}.$$

We can always do it since the series $\sum_{n=0}^{\infty} \int_{A_n} |f(x)| d\mu$ converges. Now let $0 < \delta < \frac{\epsilon}{2(N+1)}$ and $\mu(E) < \delta$ then since $|f(x)| < N+1$ on $B_N$

$$\left| \int_E f(x) d\mu \right| \leq \int_E |f(x)| d\mu = \int_{E \cap B_N} |f(x)| d\mu + \int_{E \cap C_N} |f(x)| d\mu \leq \epsilon.$$

Theorem is proved.

Using the properties of the integral proved in this section we may show that for any integrable function $f(s) \geq 0$ a set function defined on a measurable subsets $A \subset X$

$$F(A) = \int_A f(x) d\mu$$

is a $\sigma$-additive measure.

## 3.2 Passage to the limit under the Lebesgue integral

**Theorem 3.12** *(Lebesgue Dominated Convergence) Let $\{f_n(x)\}$ be a sequence of integrable functions defined on $A$, $f_n(x) \to f(x)$ a.e. $x \in A$, and for any $n$ $|f_n(x)| \leq \phi(x)$, where $\phi(x)$ is some integrable function on $A$. Then $f$ is integrable and*

$$\int_A f_n(x)d\mu \to \int_A f(x)d\mu$$

**Proof** Since $|f_n(x)| \leq \phi(x)$ and $f_n(x) \to f(x)$ a.e. we have $|f(x)| \leq \phi(x)$ a.e. and therefore $f(x)$ is an integrable function. Fix any $\epsilon > 0$, by theorem 3.11 we may find $\delta > 0$ such that $\int_B \phi(x)d\mu < \frac{\epsilon}{2}$ if $B \subset A$ and $\mu(B) < \delta$. For this particular $\delta$, using Egoroff's theorem 2.13, we may find $E_\delta \subset A$ such that $\mu(E_\delta) < \delta$ and $f_n \to f$ uniformly on $A \backslash E_\delta$. Now we have

$$\lim_{n\to\infty} \left| \int_A f_n(x)d\mu - \int_A f(x)d\mu \right| \leq \lim_{n\to\infty} \int_{A\backslash E_\delta} |f_n(x) - f(x)|d\mu$$
$$+2\int_{E_\delta} \phi(x)d\mu < \epsilon$$

Since $\epsilon$ was arbitrary we take $\epsilon \to 0$ and obtain the result.

**Theorem 3.13** *(Monotone Convergence) Let $f_1(x) \leq f_2(x) \leq ...$ be a sequence of integrable functions on $A$ and*

$$\int_A f_n(x)d\mu \leq C \quad \textit{for all } n,$$

*where $C$ is some constant independent of $n$. Then $f_n(x)$ converges a.e. on $A$ to some integrable function $f(x)$ and*

$$\int_A f_n(x)d\mu \to \int_A f(x)d\mu$$

**Proof** Without loss of generality we may assume $f_1(x) \geq 0$. We want to prove that $f_n(x) \to f(x)$ a.e. Since $f_n(x)$ is a monotone increasing sequence it is obvious that for every $x \in A$ $f_n(x) \to f(x)$ but here the value of $f(x)$ may be infinite. So our first task is to show that $f(x)$ is infinite only on some null set. We define $R = \{x \in A : \lim_{n\to\infty} f_n(x) = \infty$, $R_n^k = \{x \in A : f_n(x) > k\}$. It is easy to see that $R_1^k \subset R_2^k \subset ...$ and $R = \cap_{k=1}^\infty \cup_{n=1}^\infty R_n^k$. Using Chebyshev inequality we obtain

$$\mu(R_n^k) \leq \frac{1}{k}\int_A f_n(x)d\mu \leq \frac{C}{k} \quad \text{for any } n.$$

Now we have $\mathbb{R} \subset \cup_{n=1}^{\infty} R_n^k$ for any $k$ and therefore

$$\mu(R) \leq \mu(\cup_{n=1}^{\infty} R_n^k) = \lim_{n\to\infty} \mu(R_n^k) \leq \frac{C}{k}$$

for any $k \in \mathbb{N}$. Taking $k \to \infty$ we obtain $\mu(R) = 0$. This proves that monotone sequence $\{f_n(x)\}$ has a finite limit $f(x)$ a.e. on $A$.

Now we want to show integrability of $f(x)$. If we show this then using Lebesgue dominated convergence theorem 3.12 we get the result since $0 \leq f_n(x) \leq f(x)$ and $f_n(x) \to f(x)$ a.e. To show integrability of $f$ we construct an auxiliary function $\phi(x)$:

$$\phi(x) = k \text{ on } A_k \equiv \{x \in A \,:\, k-1 \leq f(x) < k\}.$$

It is obvious that $\phi(x)$ is elementary and $f(x) \leq \phi(x) \leq f(x)+1$. By definition $\int_A \phi(x)d\mu$ exists if and only if $\sum_{k=1}^{\infty} k\mu(A_k)$ converges. We define $B_m = \cup_{k=1}^{m} A_k$, obviously $\sum_{k=1}^{m} k\mu(A_k) = \int_{B_m} \phi(x)d\mu \leq \int_{B_m} f(x)d\mu + \mu(A)$. Since $0 \leq f(x) \leq m$ on $B_m$ by theorem 3.12 we have $\int_{B_m} f(x)d\mu = \lim_{n\to\infty} \int_{B_m} f_n(x)d\mu \leq C$.Therefore

$$\sum_{k=1}^{m} k\mu(A_k) \leq C + \mu(A)$$

and taking $m \to \infty$ we see that this series converges and $\phi(x)$ is integrable. Since $0 \leq f(x) \leq \phi(x)$ the theorem is proved.

**Theorem 3.14** *(Fatou lemma) If a sequence of non-negative integrable functions $\{f_n(x)\}$ converges a.e. on $A$ to a function $f(x)$ and*

$$\int_A f_n(x)d\mu \leq C \qquad \textit{for all } n,$$

*where $C$ is some constant independent of $n$. Then $f(x)$ is integrable on $A$ and*

$$\liminf \int_A f_n(x)d\mu \geq \int_A f(x)d\mu$$

**Proof** We prove this result using Monotone convergence theorem 3.13. We define $\phi_n(x) = \inf_{k\geq n} f_k(x)$, it is easy to see that

1. $\phi_n(x)$ is measurable for all $n$;
2. $0 \leq \phi_n(x) \leq f_n(x)$ and hence $\phi_n(x)$ is integrable for all $n$ with $\int_A \phi_n(x)d\mu \leq \int_A f(x)d\mu \leq C$;

3. $0 \leq \phi_1(x) \leq \phi_2(x) \leq ...$ and $\phi_n(x) \to f(x)$ a.e.

Using Monotone convergence theorem 3.13 we have $\lim_{n\to\infty} \int_A \phi_n(x)d\mu = \int_A f(x)d\mu$ and hence

$$\liminf \int_A f_n(x)d\mu \geq \liminf \int_A \phi_n(x)d\mu = \int_A f(x)d\mu.$$

The theorem is proved.

## 3.3 Product measures and Fubini theorem

**Definition 3.15** *The set of ordered pairs $(x_1, ..., x_n)$, where $x_i \in X_i$ for $i = 1, .., n$ is called a product of sets $X_1, ..., X_n$ is denoted by $X \equiv X_1 \times X_2 \times ... \times X_n \equiv \times_{k=1}^n X_k$.*

In particular, if $X_1 = X_2 = ... = X_n$ then $X \equiv X^n$

**Definition 3.16** *If $\mathfrak{S}_1, ..., \mathfrak{S}_n$ are collection of subsets of sets $X_1, ..., X_n$, respectively, then*

$$\mathfrak{S} \equiv \mathfrak{S}_1 \times ... \times \mathfrak{S}_n \equiv \times_{k=1}^n \mathfrak{S}_k$$

*is the collection of subsets of $X = \times_{k=1}^n X_k$ representable in the form $A = A_1 \times ... \times A_n$, where $A_k \in \mathfrak{S}_k$.*

**Theorem 3.17** *If $\mathfrak{S}_1, ..., \mathfrak{S}_n$ are semi-rings then $\mathfrak{S} = \times_{k=1}^n \mathfrak{S}_k$ is a semi-ring.*

**Proof** The proof of this theorem is left as an exercise.

**Definition 3.18** *Let $\mu_1, ..., \mu_n$ be some measures defined on the semi-rings $\mathfrak{S}_1, ..., \mathfrak{S}_n$. Then the set function*

$$\mu = \mu_1 \times ... \times \mu_n$$

*on a semi-ring $\mathfrak{S} = \mathfrak{S}_1 \times ... \times \mathfrak{S}_n$ is defined as*

$$\mu(A) = \mu_1(A_1)\mu_2(A_2) \cdots \mu_n(A_n)$$

*for $\mathfrak{S} \ni A = A_1 \times ... \times A_n$*

**Proposition 3.19** *The set function $\mu$ from definition 3.18 is a measure.*

**Proof** The proof of this proposition is left as an exercise.

**Theorem 3.20** *If the measures $\mu_1, ..., \mu_n$ are $\sigma$-additive then the measure $\mu = \times_{k=1}^{n} \mu_k$ is $\sigma$-additive.*

**Proof** The proof of this theorem is left as an exercise.

For simplicity of the presentation we consider the case $n = 2$ only. We assume that $X$ and $Y$ are some sets, $\mu_x$ and $\mu_y$ are Lebesgue measures on these sets. We also introduce $\mu = \mu_x \otimes \mu_y$ which is Lebesgue extension of a measure $m = \mu_x \times \mu_y$ on $X \times Y$.

**Definition 3.21** *Let $A \subset Z = X \times Y$ then*

$$A_x = \{y \in Y \,:\, (x, y) \in A\}, \quad A_y = \{x \in X \,:\, (x, y) \in A\}.$$

**Theorem 3.22** *Under the above assumptions on $X$, $Y$, $\mu_x$, $\mu_y$ and $\mu$ we have*

$$\mu(A) = \int_Y \mu_x(A_y) d\mu_y = \int_X \mu_y(A_x) d\mu_x$$

**Proof** We are going to prove only first equality

$$\mu(A) = \int_Y \phi_A(y) d\mu_y,$$

where $\phi_A(y) = \mu_x(A_y)$, since the second one can be done by the same arguments. By definition of $\mu$ it is Lebesgue extension of $m = \mu_x \times \mu_y$ defined on the collection of sets $\mathfrak{S}_m$ of the form $A = A_{y_0} \times A_{x_0}$, where $A_{y_0}$ is $\mu_x$-measurable and $A_{y_0}$ is $\mu_y$-measurable. For such sets $A$ we obviously have

$$\mu(A) = \mu_x(A_{y_0})\mu_y(A_{x_0}) = \int_{A_{x_0}} \mu_x(A_{y_0}) d\mu_y = \int_Y \phi_A(y) d\mu_y,$$

where

$$\phi_A(y) = \begin{cases} \mu_x(A_{y_0}) & \text{if } y \in A_{x_0}, \\ 0 & \text{otherwise .} \end{cases}$$

Note that if you make a section of such $A$ at any point $y \in Y$, you obtain either $\varnothing$ if $y \notin A_{x_0}$ or $A_{y_0}$ if $y \in A_{x_0}$. This means the theorem is true for such "rectangles" $A$. The generalization of the result to a finite disjoint union of such sets is not difficult. Since those sets coincide with $\mathfrak{R}(\mathfrak{S}_m)$ we have the theorem for this algebra.

**Lemma 3.2** *If $A$ is $\mu$-measurable set, then there exists a set $B$ such that*

$$B = \cap_n B_n, \quad B_1 \supset B_2 \supset ...,$$

$$B_n = \cup_k B_{nk}, \quad B_{n1} \subset B_{n2} \subset ...,$$

*where the sets $B_{nk} \in \mathfrak{R}(\mathfrak{S}_m)$, $A \subset B$ and*

$$\mu(A) = \mu(B).$$

The proof of this lemma is left as an exercise.
Since we can prove the theorem for any set in $\mathfrak{R}(\mathfrak{S}_m)$ and using the above lemma approximate any measurable $A$ by the special set $B \in \mathfrak{P}(\mathfrak{S}_m)$. We first prove the theorem for this $B$:

$$\phi_{B_n}(y) = \lim_{k\to\infty} \phi_{B_{nk}}(y), \text{ as } \phi_{B_{n1}}(y) \le \phi_{B_{n2}}(y) \le ...$$

$$\phi_B(y) = \lim_{k\to\infty} \phi_{B_k}(y), \text{ as } \phi_{B_1}(y) \le \phi_{B_2}(y) \le ...$$

Since we know that $\int_Y \phi_{B_{nk}}(y) d\mu_y = \mu(B_{nk})$, by continuity of $\mu$ we obtain $\mu(B_{nk}) \to \mu(B_n)$. On the other hand we have

$$\phi_{B_n}(y) = \lim_{k\to\infty} \phi_{B_{nk}}(y) \text{ and } \int_Y \phi_{B_{nk}}(y) d\mu_y \le \mu(B_n)$$

Using monotone convergence theorem we have

$$\int_Y \phi_{B_{nk}}(y) d\mu_y \to \int_Y \phi_{B_n}(y) d\mu_y = \mu(B_n).$$

By the same arguments $\int_Y \phi_{B_n}(y) d\mu_y \to \int_Y \phi_B(y) d\mu_y = \mu(B)$. This proves the theorem for this special set $B$.

Now we prove the theorem for any null set. If $\mu(A) = 0$ then by lemma $\mu(B) = 0$ and therefore

$$\int_Y \phi_B(y) d\mu_y = \mu(B) = 0.$$

But since $\phi_B(y) \ge 0$ a.e this implies $\mu_x(B_y) = \phi_B(y) = 0$ a.e. Since $A_y \subset B_y$ we have $A_y$ is measurable for almost all $y \in Y$ and

$$\phi_A(y) = \mu_x(A_y) = 0, \quad \int_Y \phi_A(y) d\mu_y = 0 = \mu(A).$$

The theorem holds for null sets. Since by the above lemma any measurable set $A = B \backslash N$ we have the result.

**Theorem 3.23** *The Lebesgue integral of a nonnegative integrable function $f(x)$ is equal to the measure $\mu = \mu_x \otimes \mu_y$ of the set*

$$A = \begin{cases} x \in M, \\ 0 \leq y \leq f(x). \end{cases}$$

**Proof** The proof is left as an exercise.

**Theorem 3.24** *(Fubini) Suppose that $\sigma$-additive and complete measures $\mu_x$ and $\mu_y$ are defined on Borel algebras with units $X$ and $Y$, respectively; further suppose that*

$$\mu = \mu_x \otimes \mu_y,$$

*and that the function $f(x, y)$ is $\mu$-integrable on $A \subset X \times Y$. Then*

$$\int_A f(x,y) d\mu = \int_X \left( \int_{A_x} f(x,y) d\mu_y \right) d\mu_x = \int_Y \left( \int_{A_y} f(x,y) d\mu_x \right) d\mu_y.$$

**Proof** We prove the theorem first for the case $f(x, y) \geq 0$. Let us consider the triple product

$$U = X \times Y \times \mathbb{R},$$

and the product measure

$$\lambda = \mu_x \otimes \mu_y \otimes \mu_1 = \mu \otimes \mu_1,$$

where $\mu_1$ is $1 - D$ Lebesgue measure. We define a set $W \subset U$ as follows:

$$W = \{(x, y, z) \in U \,:\, x \in A_x, y \in A_y, 0 \leq z \leq f(x, y)\}.$$

By theorem 3.23

$$\lambda(W) = \int_A f(x,y) d\mu,$$

on the other hand by theorem 3.22

$$\lambda(W) = \int_X \nu(W_x) d\mu_x,$$

where $\nu = \mu_y \otimes \mu_1$ and $W_x = \{(y, z) \,:\, (x, y, z) \in W\}$. But by theorem 3.23

$$\nu(W_x) = \int_{A_x} f(x,y) d\mu_y.$$

Therefore we obtain

$$\int_A f(x,y)d\mu = \int_X \left( \int_{A_x} f(x,y)d\mu_y \right) d\mu_x.$$

The theorem is proved for $f(x,y) \geq 0$. The general case reduces to this one by $f(x,y) = f^+(x,y) - f^-(x,y)$.

## Problems

1. If $f(x)$ is a measurable function, $g(x)$ is an integrable function and $\alpha, \beta \in \mathbb{R}$ are such that $\alpha \leq f(x) \leq \beta$ a.e., then there exists $\gamma \in \mathbb{R}$ such that $\alpha \leq \gamma \leq \beta$ and
$$\int_X f(x)|g(x)|d\mu = \gamma \int_X |g(x)|d\mu.$$

2. If $\{f_n(x)\}$ is a sequence of integrable functions such that
$$\sum_n \int_X |f_n(x)|d\mu < \infty,$$
then the series $\sum_n f_n(x) \to f(x)$ a.e., where $f$ is integrable and
$$\sum_n \int_X |f_n(x)|d\mu = \int_X f(x)d\mu.$$

3. Suppose $\mu = \mu_x \otimes \mu_y$ is a product measure on $X \times Y$. Show that if $f$ is $\mu$-measurable and $\int_X (\int_{A_x} |f(x,y)|d\mu_y)d\mu_x$ exists then $f$ is $\mu$-integrable on $X \times Y$ and Fubini's theorem holds.

4. Let $f \in L^1(X)$, $g \in L^1(Y)$ and $h(x,y) = f(x)g(y)$ a.e. $(x,y) \in \Omega = X \times Y$. Prove that $h \in L^1(\Omega)$ and
$$\int_\Omega h(x,y)d\mu = \int_X f(x)d\mu_x \int_Y g(y)d\mu_y.$$

5. Construct Lebesgue integral using **simple** functions.

6. Show that a space of integrable functions is complete with respect to metric
$$d(f,g) = \int_X |f(x) - g(x)|d\mu.$$
.

7. Compare Lebesgue and Riemann integral. What is the main difference in the construction and properties of these integrals?

8. Let $X = Y = [0,1]$ and $\mu = \mu_x \otimes \mu_y$, where $\mu_x = \mu_y$ is Lebesgue measure. Let $f(x), g(x)$ be integrable over $X$. If

$$F(x) = \int_{[0,x]} f(x) d\mu_x, \quad G(x) = \int_{[0,x]} g(x) d\mu_x$$

for $x \in [0,1]$, then

$$\int_X F(x) g(x) d\mu_x = G(1)F(1) - \int_X f(x) G(x) d\mu_x.$$