Extensions of Borovoi's theorem

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Chapter 1

Introduction

Sophus Lie almost single-handedly began the theory of Lie groups as it is now known, 1873. He intended to parallel the work of Évariste Galois' theory of field extensions for differentiable manifolds. The main results of his work with F. Engel are today given by three theorems and their converses. Many authors refer to these as Lie's first, second and third theorems. They allow passage between certain transformation groups and their corresponding family of 'infinitesimal transformations', now known as a Lie group and its corresponding Lie algebra.
Lie did not work directly with any of the classical groups, like SL_n, instead studying them as groups of transformations.

The idea of treating such groups systematically did not really begin until the work of Weyl. Like Sophus Lie, Weyl was inspired by pioneering work in another area, in this case the extensions to representation theory made by Schur from finite groups to other groups, such as orthogonal and unitary groups. This led to many studies concerning compact connected groups between 1924 and 1926.

The next major change of focus to greet Lie's theory was that made by Élie Cartan in 1930. Firstly, he gave a proof of Lie's third theorem for global groups, rather than locally defined groups of transformations. He also wrote the first book concerning global Lie theory, which changed the emphasis of this subject, the second chapter of this 60 page book gives rise to what is now called Cartan's theorem. He also introduced the term Lie group for the first time. The term Lie algebra was first used in lecture notes by Weyl.

The first chapter covers the basics of Lie theory, introducing Lie groups, Lie algebras and providing many of the fundamental results to this area of study.

The classification of complex semisimple Lie algebras is the focus of the second chapter. The original classification was mainly due to the work of Killing and Cartan in the 1890's. Many of the results were announced by Killing, but his proofs were often missing or incomplete. Cartan, with the assistance of both Engel and in particular his thesis student Umlauf, gave the first rigourous treatment of this classification. This differs greatly from any classification proof that would be given today as it did not include a notion of a simple root and it was Weyl who introduced the lexicographical orderings needed to define simple roots. Van der Waerden provided another simplification in 1933, and another step towards the modern proof was given by Dynkin, who provided the collection of diagrams which now bear his name.

Cartan did give the original proof of what is now known as the Existence theorem, but he did so case by case. A general approach was given by Witt over 10 years later, provided that the result held for groups of rank at most 4. The first complete general approach was due to Chevalley in 1948, introducing a free Lie algebra and factoring out by some ideal. In 1966, Serre improved this method by redefining the ideal more concretely using what are now called the Serre relations. His method also simplified the proof of the Isomorphism theorem, which again was originally proved by Cartan.

Much of the material covered by chapter 3 is again due to the work of Cartan and Weyl. Cartan classified the real forms of complex simple Lie algebras and by inspection saw that each complex simple Lie algebra admits exactly one compact real form. The development of Serre relations and new proof of the Isomorphism gave a simpler proof of the existence of compact real forms. The existence and uniqueness of a

Cartan involution (up to conjugacy) is also, unsurprisingly, a work of Cartan. The Cartan decomposition on the Lie group level is due to Mostow in 1949. Cartan gave a decomposition of certain semisimple Lie groups, called the KAK decomposition. The simply connected subgroup N was introduced by Iwasawa in 1949 and the resulting decomposition was called the Iwasawa decomposition.

Jacques Tits' book "Buildings of spherical type and finite BN-pairs", was first published in 1974 and introduced spherical buildings and BN-pairs, as the name suggests. Chapter 4 follows this theory, giving two equivalent constructions of buildings. The correspondence between spherical buildings and BN-pairs is explored, before attention is turned to twin buildings and twin BN-pairs.

Categories were first introduced by Samuel Eilenberg and Saunders Mac Lane in the early 1940's, in work concerning algebraic topology.
In his 1984 paper, "Generators and relations in compact Lie groups," Mikhail Borovoi proved what is now known as Borovoi's theorem, we will present the recent extension made in [GGH09] to prove a stronger statement concerning the topology of compact Lie groups. This represents our first extension of this theorem.

V. Kac and R. Moody independently introduced Kac-Moody algebras in their respective texts [Ka68] and [Mo68]. These algebras are in some way infinite dimensional analogues of complex semisimple Lie algebras. To these algebras we can define respective Kac-Moody groups and ask whether such groups possess an analogue to Borovoi's theorem. The answer to this question is not new, however, the method of preserving information concerning the relative topologies given by [GGH09] includes this additional information.

Using the results in chapter 2, a new approach to groups of this form started. The method was to take some Dynkin diagram and classify those groups possessing the structure necessary to be considered as groups with that Dynkin diagram and when such groups are in fact entirely determined by certain small subgroups. The final chapter gives sufficient conditions on such a group to ascertain that it is in fact isomorphic (as an amalgam) to the Phan amalgam of a unitary form of some Kac-Moody group.

Borovoi's theorem and its many extensions provide one of the most useful tools in group theory, the ability to determine a group completely from certain small subgroups. Variations and improvements on these results lead to a better understanding of the exact information needed to completely determine a group of Lie type.

1.1 Manifolds

A function $f : X \to Y$ is described as smooth (written $f \in C^\infty(X)$) if and only if all partial derivatives of f exist and are continuous on the whole of X.

Definition 1.1.1 Manifolds

Let $(X, \mathcal{T})$ be a topological space. X is called a manifold (of dimension n) if and only if there exists a set of pairs $\{(U_\alpha, \phi_\alpha) \,|\, \alpha \in A\}$ such that

(i) *$\{U_\alpha \,|\, \alpha \in A\}$ is an open cover of X,*

(ii) *for each α there is a nonempty open subset V_α of $\mathbb{R}^n$, such that $\phi_\alpha : U_\alpha \to V_\alpha$ is a homeomorphism,*

(iii) *$\phi_\alpha(x) = \phi_\beta(x)$ for all $x \in U_\alpha \cap U_\beta$.*

If, in addition, the set of pairs can be chosen such that each ϕ_α is a diffeomorphism then X is called a differentiable manifold.

The set of pairs $\{(U_\alpha, \phi_\alpha) \,|\, \alpha \in A\}$ is called an atlas of the manifold. When we refer to a topological space as a manifold, we implicitly assume that its atlas is extended to include all pairs satisfying the three properties above.

Often an n-dimensional manifold is simply referred to as an n-manifold. It is an immediate consequence of the definition that if X is an n-manifold and $f : X \to Y$ is a homeomorphism then Y is an n-manifold. Moreover, if X is a differentiable manifold and f is smooth, then Y is also a differentiable manifold.

Finally, if X is a manifold and $x \in X$ then there is some connected neighbourhood U of x which is homeomorphic to a connected open subset of $\mathbb{R}^n$, but every connected open subset of $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ itself. Therefore every manifold is locally homeomorphic to $\mathbb{R}^n$ and every differentiable manifold is locally diffeomorphic to $\mathbb{R}^n$.
From this we can deduce that all manifolds are locally compact.

Example 1.1.2 Examples of manifolds

(i) $\mathbb{R}^n$ with the usual Euclidean topology, $\mathcal{E}$, is an n-manifold with open cover $\{U_1 = \mathbb{R}^n\}$ and $\iota_1 : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

(ii) Let V be an n-dimensional real vector space, then there is some vector space isomorphism $\psi : V \to \mathbb{R}^n$, so we may equip V with the topology $\mathcal{T}$ which makes ψ a homeomorphism. Therefore V is also an n-manifold with respect to the topology $\mathcal{T}$. As ψ is linear, we deduce that V is a differentiable manifold. In particular the spaces $\mathbb{C}^n$ and $\mathbb{H}^n$ (where $\mathbb{H}$ is the field of quaternions) are both differentiable manifolds (with respect to the Euclidean topology) of dimension $2n$ and $4n$ respectively.

(iii) Let $C^n := \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \max\{x_1, \dots, x_n\} = 1\}$, so C^n is the surface of an $(n+1)$ dimensional cube. C^n is an n-manifold but is not a differentiable manifold, as no open neighbourhood of $(1, \dots, 1)$ in C^n is diffeomorphic to any open subset of $\mathbb{R}^n$.

(iv) If $(\mathbb{R}^n, \mathcal{T})$ is a topological space where $\mathcal{T}$ is such that some $x \in \mathbb{R}^n$ has no open neighbourhood homeomorphic to an open subset of $(\mathbb{R}^n, \mathcal{E})$, then $\mathbb{R}^n$ is not a manifold with respect to this topology.

Proposition 1.1.3 Open subsets of manifolds

With respect to the subspace topology an open subset of an n-manifold is an n-manifold and an open subset of a differentiable n-manifold is a differentiable n-manifold.

Proof: If $Y \subseteq X$ is open in $\mathcal{T}$ then $\{(U_\alpha \cap Y) \,|\, \alpha \in A\}$ forms an open cover of Y.
Moreover as ϕ_α is a homeomorphism, it is an open map (the image of any open set is open) and hence $\phi_\alpha(U_\alpha \cap Y)$ is homeomorphic to an open subset of $\mathbb{R}^n$. Thus $\{(U_\alpha \cap Y), \phi_\alpha \,|\, \alpha \in A\}$ is an atlas for Y. If X is differentiable, then each ϕ_α is a diffeomorphism on $(U_\alpha \cap Y)$ as required. $\square$

Example 1.1.4 Closed subsets of manifolds

Closed subsets of manifolds may or may not be manifolds.

For example if $n \geq 2$, the sphere

$$S^{n-1} = \{x \in \mathbb{R}^n \,|\, |x| = 1\} \text{ is an } (n-1)\text{-manifold}$$

and proposition 1.1.3 tells us that the open set

$$B_1(0) = \{x \in \mathbb{R}^n \,|\, |x| < 1\} \text{ is an } n\text{-manifold.}$$

However, the unit ball

$$\overline{B_1(0)} = S^{n-1} \cup B_1(0) = \{x \in \mathbb{R}^n \,|\, |x| \leq 1\}$$

is not a manifold as no open neighbourhood of any x in this set with $|x| = 1$ is homeomorphic to any open subset of $\mathbb{R}^k$, for any $k \in \mathbb{N}$.

Example 1.1.5 $GL_n(\mathbb{F})$ is a differentiable manifold

$M_n(\mathbb{C})$, the set of all $n \times n$ matrices with complex entries, is a complex vector space of dimension n^2, so is a differentiable $4n^2$-manifold by example 1.1.2(ii). Consider the determinant map $\det : M_n(\mathbb{C}) \to \mathbb{C}$. As $\mathbb{C} \setminus \{0\}$ is an open subset of $\mathbb{C}$ and the determinant map is continuous, it follows that

$$GL_n(\mathbb{C}) = \{M \in M_n(\mathbb{C}) \,|\, \det(M) \neq 0\} = \det^{-1}(\mathbb{C} \setminus \{0\})$$

is an open subset of $M_n(\mathbb{C})$. Therefore, $GL_n(\mathbb{C})$ is a differentiable manifold, by proposition 1.1.3. Two similar arguments verify that $GL_n(\mathbb{R})$ and $GL_n(\mathbb{H})$ are both differentiable manifolds of dimensions n^2 and $16n^2$ respectively. □

Definition 1.1.6 Derivations and the tangent bundle

Let M be a manifold and let $p \in M$. A derivation at p is a linear map $D : C^\infty(M) \to \mathbb{C}$ which obey the Liebniz product rule

$$D(f.g)(p) = D(f)g(p) + f(p)D(g) \text{ for all } f, g \in C^\infty(M),$$

where $(f.g)(p) := f(p).g(p)$. The set of all derivations at x will be denoted by $T_x(M)$.

The tangent bundle of M is $TM = \bigsqcup_{x \in M} T_x(M)$.

Definition 1.1.7 (Smooth) vector fields

A function $X : M \to TM$ is called a (smooth) vector field if and only if $X \in C^\infty(M)$ and $X(p) \in T_p(M)$ for all $p \in M$.

Example 1.1.8 Tangent vectors

One collection of derivations are provided by curves on a manifold. A curve is simply a smooth map $\gamma : I \to M$ where $I \subseteq \mathbb{R}$ is an open interval containing 0.

Let $p \in M$ and let γ be any curve in M with $\gamma(0) = p$. As γ is smooth,

$$\frac{d}{dt}\gamma(t)\,|_{t=0} = \gamma'(0) \text{ is a well defined vector in } \mathbb{R}^n.$$

$\gamma'(0)$ is called a tangent vector to the manifold M at the point p.

To see how this defines a derivation, suppose $f : M \to \mathbb{R}$ is a smooth function and consider

$$Df_\gamma = \frac{d}{dt} f(\gamma(t))\,|_{t=0} = \gamma'(0)f'(\gamma(0))$$

As $\gamma(0) = p$ is fixed, Df_γ depends only on the tangent vector $\gamma'(0)$. We can immediately see from the definition that each curve passing through p defines a derivation at p, by the product rule of differentiation.

The next goal will be to prove that for a differentiable manifold all derivations at p can be obtained from the tangent vectors of curves. Before this we need to introduce the notion of a differential and state two important properties which differentiable manifolds inherit from $\mathbb{R}^n$.

Definition 1.1.9 Differentials

Let M and N be differentiable manifolds and let $f : M \to N$ be smooth. The differential of f at $p \in M$ is the linear map $df_p : T_p(M) \to T_{f(p)}(N)$ with the property that given any function $g \in C^\infty(U)$ where U is an open neighbourhood of $f(p)$ in N, the formula $df_p(v)(g) = v(g \circ f)$ holds.

It is not obvious from this definition that such a differential necessarily exists. We will show that differentials do exist and that there is a neat formula for df_p.

Theorem 1.1.10 Two key theorems for differentiable manifolds

(i) **Inverse function theorem**

Let M and N be differentiable manifolds. A map $\phi : M \to N$ is locally diffeomorphic at $p \in M$ if and only if its differential $d\phi : T_p(M) \to T_{\phi(p)}(N)$ is a vector space isomorphism.

(ii) **Existence and uniqueness theorem for ODE's**

Let X be a smooth vector field on a differentiable manifold M and let $p \in M$. There exists an open neighbourhood U of p, an open interval I containing 0 and a smooth mapping $\psi : I \times U \to M$ such that for each $q \in U$ the curve $\psi_q : I \to M$ given by $\psi_q(t) = \psi(t, q)$ is the unique curve that satisfies $\frac{\partial \psi}{\partial t} = X_{\psi_q(t)}$ subject to the initial condition $\psi_q(0) = q$.

For a proof of (i) the reader is referred to [Wa71] pages 22-29 and to [Co01], theorem 2.8.4 (pages 72-73) and appendix C (pages 379-385) for a proof of (ii). $\square$

Theorem 1.1.11 Derivations and tangent vectors

Let M be a differentiable n-manifold and let $p \in M$.

(i) *The set of all tangent vectors at p forms a vector space over $\mathbb{R}$ of dimension n.*

(ii) *$T_p(M)$ is an n dimensional real vector space.*

(iii) *D is a derivation at p if and only if there is a curve γ such that*

$$Df = \frac{d}{dt} f(\gamma(t))\,|_{t=0} \quad \text{for all } f \in C^\infty(M) \text{ and } \gamma(0) = p.$$

Proof: Let $p \in U_\alpha \subseteq M$ where U_α is the open neighbourhood of p given by the definition and ϕ_α is the corresponding diffeomorphism from U_α to $\mathbb{R}^n$. (The remarks following definition 1.1.1 show such a diffeomorphism always exists).

(i) The set of tangent vectors inherits its vector space properties from $\mathbb{R}^n$. Let γ_1 and γ_2 be curves which define tangent vectors at p and let $\lambda \in \mathbb{R}$.

To make $T_p(M)$ a vector space we define

$$(\gamma_1{}' + \gamma_2{}')(t) = \phi_\alpha^{-1}\left[\phi_\alpha(\gamma_1(t)) + \phi_\alpha(\gamma_2(t))\right] \quad \text{and} (\lambda\gamma_1{}')(t) = \phi_\alpha^{-1}\left[\lambda\phi_\alpha(\gamma_1(t))\right].$$

If γ is a curve in U_α then $\phi_\alpha \circ \gamma$ is a smooth map from $\mathbb{R}$ to $\mathbb{R}^n$, thus we associate an element of $\mathbb{R}^n$ to each tangent vector $\gamma'(0)$ as follows.

Let γ be a curve in U_α which has tangent vector $\gamma'(0)$ at p. Set $x_\gamma = \frac{d}{dt}(\phi_\alpha \circ \gamma)(0) \in \mathbb{R}^n$.

We claim that the map $T_p(M) \to \mathbb{R}^n$ given by $\gamma \mapsto x_\gamma$ is bijective, which will suffice to prove the result.

To show injectivity, suppose $\gamma_1(0) = \gamma_2(0) = p$ and

$$\frac{d}{dt}(\phi_\alpha \circ \gamma_1)(0) = \gamma_1{}'(0).\phi_\alpha{}'(\gamma_1(0)) = \gamma_2{}'(0).\phi_\alpha{}'(\gamma_2(0)) = \frac{d}{dt}(\phi_\alpha \circ \gamma_2)(0).$$

We know $\phi_\alpha{}'(\gamma_1(0)) = \phi_\alpha{}'(\gamma_2(0))$ as ϕ_α is a smooth function. Therefore $\gamma_1{}'(0) = \gamma_2{}'(0)$.

To show surjectivity, consider a connected local neighbourhood U of p. By choosing canonical local coordinate vectors e_i for i from 1 to n, it is clear that there is some $\varepsilon > 0$ such that $p + (k\varepsilon)e_i \in U$ for all $k \in (-1, 1)$ and all i.

Set $\gamma_i : (-\varepsilon, \varepsilon) \to M$ with $\gamma_i(x) = p + x.e_i$. Then $\gamma_i{}'(0) = e_i$ and therefore this vector space contains n linearly independent vectors, justifying the claim.

(ii) Every differentiable manifold is locally diffeomorphic to $\mathbb{R}^n$. Hence, for any point $p \in M$, $T_p(M) \cong T_{\phi_\alpha(p)}(\mathbb{R}^n) \cong \mathbb{R}^n$, by the inverse function theorem on manifolds (1.1.10(i)).

(iii) It is immediately clear from example 1.1.8 that every curve with the required properties defines a derivation at p by basic properties of calculus.

Consider the subspace of $T_p(M)$ which contains all derivations which have such a curve associated to them. As the dimension of $T_p(M)$ is equal to the dimension of this subspace, it follows that this subspace is the entire of $T_p(M)$, which completes the proof. □

As a result of this $T_x(M)$ is called the tangent space of M at x.

Proposition 1.1.11 provides a useful method for calculating the differential of a smooth function f at $p \in M$. Given any curve γ with the properties that $\gamma(0) = p$ and $\gamma'(0) = v$, the differential of f at p is given by

$$df_p = \frac{d}{dt} f(\gamma(t))\,|_{t=0}\ .$$

Definition 1.1.12 Connectedness properties

Let M be an n-manifold. M is said to be

(i) *connected if and only if it cannot be partitioned into two disjoint open sets,*

(ii) *path-connected if and only if given any two distinct points $x, y \in M$ there is a continuous function $f : [0, 1] \to M$ with $f(0) = x$ and $f(1) = y$,*

(iii) *simply connected if and only if it is path connected and for each continuous map $f : S^1 \to M$ there is another continuous map*

$$F : \overline{B_1(0)} \to Mz, \text{ with the property that } F|_{S^1} = f.$$

To illustrate the definition we prove that if M is path-connected then it is connected.

Suppose the converse is true and let x and y lie in two disjoint open sets U and V which partition M. By assumption, M is path-connected so there is a continuous function $f : [0, 1] \to M$ with $f(0) = x$ and $f(1) = y$.
As f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open subsets of $[0, 1]$ which partition this interval. But $[0, 1]$ is connected so this is a contradiction. □

Definition 1.1.13 Submanifolds

Let M be a manifold and let $N \subseteq M$. N is called an immersed submanifold of M if and only if

(i) *N is itself a manifold,*

(ii) *the differential of the identity map $d\iota : T_p(N) \to T_p(M)$ is injective for all $p \in N$.*

If, in addition, the topology on N is the subspace topology inherited from M, then N is called an embedded (or regular) submanifold.

1.2 Lie Groups and Lie Algebras

Definition 1.2.1 Lie groups

Let G be a group and let $\mathcal{T}$ be a topology on G. G is called a Lie group if and only if

(i) *G is a differentiable manifold,*

(ii) *$(G, \mathcal{T})$ is a smooth topological group, so the function $\theta : G \times G \to G$ given by $\theta(g, h) = gh^{-1}$ is smooth,*

(iii) *$\mathcal{T}$ is Hausdorff and second countable.*

A subgroup H of G is called a Lie subgroup of G if and only if H is an immersed submanifold of G.

As every Lie group is a differentiable manifold, so all of the results in section 1.1 apply to Lie groups. We also note that every Lie subgroup of a Lie group must be a Lie group itself.

One trivial example of a Lie group is $\mathbb{R}^n$ under addition. A more important example is the general linear group over any of the fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. A proof of this fact for $\mathbb{R}$ and $\mathbb{C}$ can be found in [Kn02], on pages 12-15.

Theorem 1.2.2 Cartan's theorem

Let G be a Lie group and let H be a subset of G equipped with the subspace topology inherited from G. H is a Lie subgroup of G if and only if it is a closed subgroup of G.

Proof: See [Co01], proposition 5.3.1 and theorem 5.3.2, pages 173-175, or alternatively [Wa71], theorem 3.21. □

Example 1.2.3 G_0 is a normal Lie subgroup of G

A simple consequence of Cartan's theorem is that given any Lie group G, the connected component of the identity, G_0, is a normal Lie subgroup of G. To show G_0 is closed we note that the closure of G_0 is itself connected, so G_0 must be closed as it is the maximal connected subset of G (by inclusion) containing the identity. It remains to show that G_0 is a normal subgroup of G.
Firstly, we note that as G is a topological group, multiplication by an element from the group is a homeomorphism (it is a continuous bijection whose inverse is multiplication by g^{-1} which is also continuous). In particular the image of a closed connected subset under group multiplication is closed and connected. Let $g, h \in G_0$, then the image of G_0 under right multiplication by h is a closed connected subset of G which contains h (as $1 \in G_0$) and gh. Therefore, $gh \in G_0$.
Similarly, the image of G_0 under right multiplication by g^{-1} contains both g^{-1} and 1, so $g^{-1} \in G_0$. By Cartan's theorem G_0 is a Lie subgroup of G.
Finally, we note that group conjugation is the composition of two group multiplication functions. Let $g \in G_0$ and $h \in G$. Then $h^{-1}G_0h$ is a closed connected subset of G which contains $h^{-1}1h = 1$ and $h^{-1}gh$, so $h^{-1}gh \in G_0$. Thus $G_0 \trianglelefteq G$.

Example 1.2.4 Important Lie subgroups of the general linear group

Returning to the example $G = GL_n(\mathbb{R})$ and using Cartan's theorem we see that

$$SL_n(\mathbb{R}) = \{M \in GL_n(\mathbb{R} \mid \det M = 1\} = \det^{-1}(\{1\})$$

is a Lie subgroup of $GL_n(\mathbb{R})$ as the determinant map is continuous and $\{1\}$ is a closed subset of $\mathbb{R}$. Hence $SL_n(\mathbb{R})$ is a Lie group. As $\det : GL_n(\mathbb{C}) \to \mathbb{C}$ ic continuous and $\{1\}$ is a closed subset of $\mathbb{C}$, we deduce that $SL_n(\mathbb{C})$ is a Lie subgroup of $GL_n(\mathbb{C})$ and therefore a Lie group.

The general linear groups have many other important Lie subgroups, those of particular interest to this project include the special unitary group SU, the special orthogonal group SO and the symplectic group Sp.

The special unitary group is the subgroup

$$SU_n(\mathbb{C}) = \{X \in GL_n(\mathbb{C}) \mid X^*X = I_n \text{ and } \det(X) = 1\},$$

where X^* denotes the conjugate transpose of X.

$SU_n(\mathbb{C}) = \psi^{-1}(I_n)$, where $\psi : SL_n(\mathbb{C}) \to M_n(\mathbb{C})$ maps X to X^*X is a continuous map, so $SU_n(\mathbb{C})$ is a closed subgroup of $SL_2(\mathbb{C})$.

The condition $X^*X = 1$ ensures that each entry of a matrix $X \in SU_n(\mathbb{C})$ has modulus at most 1. Thus, applying the Heine-Borel theorem to each entry in turn we see that $SU_n(\mathbb{C})$ is a compact subgroup of $SL_n(\mathbb{C})$.

A similar argument shows that the special orthogonal group

$$SO_n(\mathbb{R}) = \{X \in GL_n(\mathbb{R}) \,|\, X^*X = X^tX = I_n \text{ and } \det(X) = 1\}$$

and the symplectic group

$$Sp_n(\mathbb{H}) = \{X \in GL_n(\mathbb{H}) \,|\, X^*X = 1\}$$

are compact Lie subgroups of $SL_n(\mathbb{R})$ and $GL_n(\mathbb{H})$ respectively.

Definition 1.2.5 Left invariant vector fields

Let G be a Lie group, with $a \in G$ and let X be a vector field on G.

X is said to be left-invariant if and only if $X(L_a) = dL_a(X)$, where $L_a : G \to G$ is the diffeomorphism given by left multiplication by a, in other words, $L_a(g) = ag$ for all $g \in G$.
The set of all left-invariant vector fields on G is denoted by $\mathfrak{g}$.

As a shorthand, $X(p)$ will be denoted by X_p.

Left invariant vector fields have the key property that they are determined by their value at $e = 1_G$, since $X_a = dL_a(X_e)$, leading to the following proposition.

Proposition 1.2.6 Left-invariant vector fields and derivations

The function $\psi : \mathfrak{g} \to T_e(G)$ given by $\psi(X) = X_e$ is a linear vector space isomorphism.

A vector field X may be thought of as a smooth endomorphism on $C^\infty(M)$, where given some $f \in C^\infty(M)$, $Xf(p)$ is defined pointwise by $X_p(f)$ for each $p \in G$.

With this in mind, define $[X, Y] = XY - YX$. Then it is clear that $\mathfrak{g}$ is closed under the binary operation $[\cdot, \cdot]$.

Definition 1.2.7 Lie algebras

An $\mathbb{F}$ algebra is a vector space $\mathfrak{g}$ over $\mathbb{F}$ with a distributive multiplication $[\cdot, \cdot] : V \times V \to V$ which has the property $a[x, y] = [ax, y] = [x, ay]$ for all $a \in \mathbb{F}$.

Such an $\mathbb{F}$-algebra $\mathfrak{g}$ is called a Lie algebra if it satisfies the additional properties

(i) $[X, X] = 0$ *for all* $X \in \mathfrak{g}$.

(ii) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *for all* $X, Y, Z \in \mathfrak{g}$ *and all* $a, b \in \mathbb{F}$.

(iii) $\mathfrak{g}$ *satisfies the Jacobi identity,* $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *for all* $X, Y, Z \in \mathfrak{g}$.

A Lie subalgebra $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ with the property that $[\mathfrak{h}, \mathfrak{h}] = \{[X, Y] \,|\, X, Y \in \mathfrak{h}\} \subseteq \mathfrak{h}$

The multiplication $[\cdot, \cdot]$ in a Lie algebra is called the Lie bracket.

We will only consider Lie algebras when $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Under this assumption condition (i) is equivalent to requiring that $[X, Y] = -[Y, X]$ for all $X \in \mathfrak{g}$. Applying this to (ii) says that the Lie bracket is bilinear.

Example 1.2.8 Examples of Lie algebras

(i) $\mathbb{C}^n$ is a Lie algebra with respect to the multiplication $[a, b] = 0$ for all $a, b \in \mathbb{C}^n$. A Lie algebra whose Lie bracket always evaluates to 0 is called abelian.

(ii) The $\mathbb{F}$-algebra of all $n \times n$ matrices, $M_n(\mathbb{F})$, is a Lie algebra with respect to the Lie bracket $[X, Y] = XY - YX$.

(iii) Any $\mathbb{F}$-algebra of smooth functions from $\mathbb{F}$ to itself is a Lie algebra with respect to the Lie bracket $[f, g](x) = (fg - gf)(x) = f(x)g(x) - g(x)f(x)$.

We give the following important examples of matrix Lie algebras.

The Lie algebras $\mathfrak{sl}_n(\mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \ \mathrm{Tr}(X) = 0\}$ is a Lie subalgebra of $M_n(\mathbb{R})$ as

$$\mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(AB) = 0.$$

Similarly, $\mathfrak{sl}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}) \mid \ \mathrm{Tr}(X) = 0\}$ is a Lie subalgebra of $M_n(\mathbb{C})$.

Next, we introduce two more Lie algebras which will be important examples in chapter 2.

$$\mathfrak{so}_n(\mathbb{F}) := \{X \in M_n(\mathbb{F}) \,|\, X + X^t = 0\}, \text{ for } n \geq 3 \text{ and}$$

$$\mathfrak{sp}_n(\mathbb{C}) := \{X \in M_{2n}(\mathbb{C}) \,|\, X^t J + JX = 0\} \text{ for } n \geq 1, \ \text{ where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \text{ is a } 2n \times 2n \text{ matrix.}$$

Notice that for $n \geq 2$, $\mathfrak{so}_n(\mathbb{F}) \subsetneq \mathfrak{sl}_n(\mathbb{F})$, as

$$0 = \mathrm{Tr}(0) = \mathrm{Tr}(X + X^t) = \mathrm{Tr}(X) + \mathrm{Tr}(X^t) = 2\mathrm{Tr}(X), \ \text{ so } \mathrm{Tr}(X) = 0.$$

Finally, we present the real Lie algebra $\mathfrak{su}_n(\mathbb{C}) := \{X \in M_n(\mathbb{C}) \,|\, X + X^* = 0 \ \text{ and } \ \mathrm{Tr}(X) = 0\}$.

Again, $\mathfrak{su}_n(\mathbb{C}) \subsetneq \mathfrak{sl}_n(\mathbb{C})$.

Theorem 1.2.9 The Lie algebra of a Lie group

Let G be a Lie group and let $\mathfrak{g}$ be the set of left-invariant vector fields on G. Then $\mathfrak{g}$ is a Lie algebra with respect to the bracket operation $[X, Y] = XY - YX = X_p(Y(f)) - Y_p(X(f))$.

Proof: There are a number of properties to check:

Firstly, if X, Y are left invariant vector fields then so is $aX + bY$ for all $a, b \in \mathbb{F}$ as TG is an $\mathbb{F}$ vector space and differentials are linear.

$[X, X] = XX - XX = 0$, and $\mathfrak{g}$ is closed under bracket multiplication as

$$\begin{aligned} dL_a[X,Y]_p f &= [X,Y]_p(f \circ L_a) = X_p(Y(f \circ L_a)) - Y_p(X(f \circ L_a)) \\ &= X_p(dL_a(Y))f - Y_p(dL_a(X))f = X_pY(f) - Y_p(X(f)) = [X,Y]_p(f). \end{aligned}$$

Next we show that the bracket is linear in the first variable

$$\begin{aligned} [aX + bY, Z] &= (aX + bY)Z - Z(aX + bY) = aXZ + bYZ - aZX - bZY \\ &= a(XZ - ZX) + b(YZ - ZY) = a[X, Z] + b[Y, Z]. \end{aligned}$$

Finally we verify that the Jacobi identity holds,

$$\begin{aligned} [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] &= (XYZ - XZY - YZX + ZYX) \\ &\quad +(YZX - YXZ - ZXY + XZY) \\ &\quad +(ZXY - ZYX - XYZ + YXZ) \\ &= 0. \end{aligned}$$

$\mathfrak{g}$ is called the Lie algebra of G. By proposition 1.2.6, $T_e(G)$ is isomorphic to $\mathfrak{g}$ when equipped with the Lie bracket $[u, v] = [X^u, X^v]_e$ where the vector field X^u is defined by $X^u_g = (dL_g)_e(u)$.

Example 1.2.10 The Lie algebras of $GL_n(\mathbb{F})$ and $SL_n(\mathbb{F})$

In the following example we define $E_{ij} = (y_{mn})$ to be the matrix given by $y_{mn} = \begin{cases} 1, & \text{if } i = m \text{ and } j = n \\ 0, & \text{otherwise} \end{cases}$

Recall that we are only considering the cases $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Let $X \in GL_n(\mathbb{F})$ and notice that there is some $\varepsilon > 0$ such that $X + t.E_{ij} \in GL_n(\mathbb{F})$ for all i, j and all $t \in (-\varepsilon, \varepsilon)$.

Hence the curves $\gamma_{ij} : (-\varepsilon, \varepsilon) \to GL_n(\mathbb{F})$ given by $\gamma_{ij}(t) = X + t.E_{ij}$ all have the properties $\gamma_{ij}(0) = X$ and $\gamma'_{ij}(0) = E_{ij}$.

As the tangent space is an $\mathbb{F}$ vector space this is sufficient to prove that $T_X(GL_n(\mathbb{F})) = M_n(\mathbb{F})$.

For the example $SL_n(\mathbb{F})$, with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we use a slight variation on the method given in [Kn02].

The key result needed here is that given a curve $\gamma : (-\varepsilon, \varepsilon) \to M_n(\mathbb{F})$ with the property that $\gamma(0) = I_n$, then

$$\frac{d}{dt}\left[\det \gamma(t)\right]_{t=0} = \mathrm{Tr}(\gamma'(0)).$$

To prove this result we write the curve as a set of matrix coordinates, $\gamma(t) = (\gamma_{ij}(t))$ then expand the determinant along the top row.

$$\det \gamma(t) = \sum_{i=1}^{n} (-1)^{n-1} \gamma_{1i}(t) \det \gamma_i(t)$$

where $\gamma_i(t)$ is the $(n-1) \times (n-1)$ matrix remaining from $\gamma(t)$ when the first row and ith column are removed.

Applying the product rule to each term in the sum gives

$$\gamma_{1i}{}'(0) \det \gamma_i(0) + \gamma_{1i}(0). \frac{d}{dt}\left[\det \gamma_i(t)\right]_{t=0}.$$

If $i \neq 1$ then $\det \gamma_i(0) = \gamma_{1i}(0) = 0$, therefore

$$\frac{d}{dt}\left[\det \gamma(t)\right]_{t=0} = \gamma_{11}{}'(0) + \frac{d}{dt}\left[\det \gamma_1(t)\right]_{t=0}.$$

Repeating this process gives the required result.

Using this fact we can deduce that the Lie algebra of $SL_n(\mathbb{F})$ is $\mathfrak{sl}_n(\mathbb{F})$.

Let $\gamma : I \to SL_n(\mathbb{F})$ be a curve with $\gamma(0) = I$. Then $\mathrm{Tr}(\gamma'(0)) = 0$ as the determinant is constant across the curve.

To show every element of $\mathfrak{sl}_n(\mathbb{F})$ is contained in the tangent space consider the curves

$$\gamma_{ij} : (-1, 1) \to SL_n(\mathbb{F}) \text{ given by } \gamma_{ij}(t) = I_n + t.E_{ij} \text{ for all } i \neq j.$$

$\gamma'_{ij}(0) = E_{ij}$ so as the tangent space is a vector space, all possible non-diagonal entries are accounted for. As a result it remains to deduce that given any diagonal matrix with trace 0, there is a suitable curve. Let $A = (a_{ij})$ be a diagonal matrix with trace 0. Define

$$\gamma_A : (-1, 1) \to SL_n(\mathbb{F}) \text{ such that } \gamma_A(t) = (b_{ij}) \text{ where } b_{ij} = \begin{cases} e^{a_{ij}t}, & \text{whenever i=j,} \\ b_{ij} = 0, & \text{otherwise.} \end{cases}$$

Then $\gamma'_A(0) = A$ and the example is complete.

Without proof we state here that the Lie algebras of $SU_n(\mathbb{C})$ and $SO_n(\mathbb{R})$ are $\mathfrak{su}_n(\mathbb{C})$ and $\mathfrak{so}_n(\mathbb{R})$ respectively.

Definition 1.2.11 One parameter subgroups

Let G be a Lie group. A one parameter subgroup of G is a smooth group homomorphism $\phi : (\mathbb{R}, +) \to G$.

Note that a one-parameter subgroup is simply a curve with the additional properties

(i) $\phi(s + t) = \phi(s)\phi(t)$,

(ii) $\phi(0) = e$ and

(iii) $\phi(-t) = \phi(t)^{-1}$.

The derivative of a one parameter subgroup is given by

$$\phi'(t) = \lim_{h\to 0} \frac{1}{h}\left[\phi(t+h) - \phi(t)\right] = \phi(t) \lim_{h\to 0} \frac{1}{h}\left[\phi(h) - \phi(0)\right] = A\phi(t),$$

where $A = \lim_{h\to 0} \frac{1}{h}\left[\phi(h) - \phi(0)\right]$.

This limit exists because G is a differentiable manifold.

Theorem 1.2.12 One parameter subgroups and tangent spaces

The map $\phi \mapsto d\phi(1)$ defines an isomorphism between the set of all one parameter subgroups of G and $T_e(G)$.

Proof: Let $v \in T_e(G)$ and set $X_g(v) = [dL_g(e)](v)$. Note X_g is a smooth left invariant vector field. By proposition 1.1.10(ii), let $\phi : I \to G$ be the unique curve with the properties $\phi(0) = e$ and $d\phi_t = X^v_{\phi(t)}$.

The curve ϕ is a homomorphism as $\phi_t(s) = \phi(s+t)$ and $\phi^t(s) = \phi(s)\phi(t)$ both satisfy the two properties above, so by uniqueness, $\phi(s+t) = \phi(s)\phi(t)$. ϕ can be extended to a one parameter subgroup $\phi_v : \mathbb{R} \to G$ by defining $\phi_v(t) = \phi(\frac{t}{n})^n$ for suitably large n.
As ϕ is a homomorphism, $\phi(\frac{t}{n})^n = \phi(\frac{t}{m})^m$ for all $m, n \in \mathbb{N} \setminus \{0\}$, so ϕ_v is a well defined homomorphism. Moreover, ϕ_v is smooth because ϕ is smooth and the inverse map is given by $v \mapsto \phi_v$, so this map is a bijection. □

The following corollary is a direct consequence of the method used in the previous proof.

Corollary 1.2.13 *For each $X \in \mathfrak{g}$ there is a unique one parameter subgroup $\phi_X : \mathbb{R} \to G$ such that $\phi'_X(0) = X$.*

This corollary allows us to define a map which will relate Lie algebras to Lie groups.

Definition 1.2.14 The exponential map

Let G be a Lie group and let $\mathfrak{g}$ be the Lie algebra of G. The map $\exp : \mathfrak{g} \to G$ defined by $\exp(X) = \phi_X(1)$ is called the exponential map.

Example 1.2.15 The Lie groups of $M_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F})$.

To calculate the exponential map for a Lie algebra of matrices, it suffices to find the unique curve $\phi : I \to GL_n(\mathbb{F})$ with the properties

$$\phi(0) = I \text{ and } d\phi_t = \frac{d}{dt}\left[\phi(t)\right]_{t=0} = X\phi(t).$$

The next proposition states that the matrix exponential function

$$t \mapsto e^{tX} = \sum_{i=0}^{\infty} \frac{X^i}{i!}$$

satisfies these properties.

This will prove the unsurprising result, that the exponential function for matrix groups is, in fact, the exponential function.

Proposition 1.2.16 Properties of the matrix exponential function

Let $X, Y \in M_n(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

(i) $e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$,

(ii) *If $XY = YX$ then $e^X e^Y = e^{X+Y}$,*

(iii) *$t \mapsto e^{tX}$ is a smooth curve with $0 \mapsto I_n$,*

(iv) $\frac{d}{dt}(e^{tX}) = Xe^{tX}$,

(v) $\det(e^X) = e^{\mathrm{Tr}(X)}$, *in particular, $e^X \in GL_n(\mathbb{F})$.*

Proof: See [Kn02] pages 7-8. □

From part (iv) of this proposition we deduce that the Lie groups of $M_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F})$ are Lie subgroups of $GL_n(\mathbb{F})$ and $SL_n(\mathbb{F}$ respectively.

1.3 Properties of Lie algebra

Definition 1.3.1 Homomorphisms

Let G, H be Lie groups and let $\mathfrak{g}, \mathfrak{h}$ be their respective Lie algebras.

A map $\phi : G \to H$ is called a Lie group homomorphism if and only if it is a smooth group homomorphism.

A map $\psi : \mathfrak{g} \to \mathfrak{h}$ is called a Lie algebra homomorphism if and only if it is a vector space homomorphism with the property $\psi([X,Y]) = [\psi(X), \psi(Y)]$ for all $X, Y \in \mathfrak{g}$.

As should be expected, an injective homomorphism (of either structure) is called a monomorphism, a surjective homomorphism is called an epimorphism and a bijective homomorphism is called an isomorphism.

If $H = G$ then a Lie group homomorphism $\phi : G \to G$ is called an endomorphism and an isomorphism is called an automorphism.
Similarly, if $\mathfrak{h} = \mathfrak{g}$ then a Lie algebra homomorphism is called an endomorphism and a bijective endomorphism is called an automorphism.

The sets $\mathrm{End}(G)$ and $\mathrm{Aut}(G)$, $\mathrm{End}(\mathfrak{g})$ and $\mathrm{Aut}(\mathfrak{g})$ consist of all endomorphisms and automorphisms on a Lie group G or Lie algebra $\mathfrak{g}$ respectively.

Example 1.3.2 The conjugation automorphism

The map $I_g : G \to G$ given by $I_g(h) = g^{-1}hg$ is a smooth (and hence continuous) automorphism of G. This is obvious since group multiplication is a smooth automorphism in a Lie group and the composition of two smooth automorphisms is itself a smooth automorphism.

Proposition 1.3.3 Relating homomorphisms

Let G, H be Lie groups and let $\mathfrak{g}, \mathfrak{h}$ be their respective Lie algebras and suppose $\phi : G \to H$ is a Lie group homomorphism. Then

(i) *The differential $d\phi_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

(ii) $\phi(\exp(X)) = \exp(d\phi_e(X))$.

Proof: For part (i) see [Wa71] theorem 3.14(b).

Let $X \in \mathfrak{h}$. The map $t \mapsto \phi(\exp(tX))$ is a smooth curve in G whose tangent at 0 is $d\phi_e(X)$. Since ϕ is a homomorphism this map extends uniquely to a 1-parameter subgroup of G.

However, the map $t \mapsto \exp(t(d\phi_e(X)))$ is also a 1-parameter subgroup of G whose tangent at 0 is $d\phi_e(X)$. By corollary 1.2.13,

$$\phi(\exp(tX)) = \exp(t(d\phi_e(X))) \quad \text{for all } t \in \mathbb{R}.$$

Therefore, $\phi(\exp(X)) = \exp(d\phi_e(X))$. □

Definition 1.3.4 Ideals of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$. $\mathfrak{h}$ is called an ideal of $\mathfrak{g}$ if and only if $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

Example 1.3.5 Some basic ideals

(i) $\{0\}$ and $\mathfrak{g}$ are always ideals of any Lie algebra $\mathfrak{g}$.

(ii) If $\mathfrak{a}$ is an abelian Lie algebra, then all its subalgebras are ideals, as the bracket is equal to the zero map.

(iii) $\mathfrak{sl}_n(\mathbb{F})$ is an ideal in $M_n(\mathbb{F})$, since $\mathrm{Tr}[A, B] = 0$ for all $A, B \in M_n(\mathbb{F})$.

Proposition 1.3.6 Constructing ideals

Given two ideals $\mathfrak{a}, \mathfrak{b}$ of a Lie algebra $\mathfrak{g}$, the following are also ideals.

(i) $\mathfrak{a}+\mathfrak{b}$,

(ii) $\mathfrak{a}\cap\mathfrak{b}$ *and*

(iii) $[\mathfrak{a},\mathfrak{b}]$.

Proof:

(i) Let $X\in\mathfrak{a}$, $Y\in\mathfrak{b}$ and $Z\in\mathfrak{g}$, then $[X+Y,Z]=[X,Z]+[Y,Z]\in\mathfrak{a}+\mathfrak{b}$.

(ii) If $X\in\mathfrak{a}\cap\mathfrak{b}$, then $[X,Z]\in\mathfrak{a}$ and $[X,Z]\in\mathfrak{b}$ by definition.

(iii) By the Jacobi identity $[[\mathfrak{a},\mathfrak{b}],\mathfrak{g}]\subseteq-[[\mathfrak{b},\mathfrak{g}],\mathfrak{a}]-[[\mathfrak{g},\mathfrak{a}],\mathfrak{b}]$.

Using antisymmetry, $[[\mathfrak{a},\mathfrak{b}],\mathfrak{g}]\subseteq[\mathfrak{a},[\mathfrak{b},\mathfrak{g}]]+[[\mathfrak{a},\mathfrak{g}],\mathfrak{b}]\subseteq[\mathfrak{a},\mathfrak{b}]+[\mathfrak{a},\mathfrak{b}]\subseteq[\mathfrak{a},\mathfrak{b}]$. □

Theorem 1.3.7 Relating ideals and homomorphisms

$\mathfrak{a}$ *is an ideal of* $\mathfrak{g}$ *if and only if there is some Lie algebra homomorphism* $\psi:\mathfrak{g}\to\mathfrak{h}$ *with* $\ker(\psi)=\mathfrak{a}$.

Proof: Let $X\in\ker(\psi)$ and let $Y\in\mathfrak{g}$. Then $\psi[X,Y]=[\psi(X),\psi(Y)]=[0,\psi(Y)]=0$, so $[X,Y]\in\ker(\psi)$.

If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ then the quotient Lie algebra $\mathfrak{g}/\mathfrak{a}$ is the Lie algebra given by the operations

$$(X+\mathfrak{a})+(Y+\mathfrak{a})=X+Y+\mathfrak{a}\quad\text{and}\quad[X+\mathfrak{a},Y+\mathfrak{a}]=[X,Y]+\mathfrak{a}.$$

Using this we define the quotient map $\psi:\mathfrak{g}\to\mathfrak{g}/\mathfrak{a}$ by $\psi(X)=X+\mathfrak{a}$ and notice that ψ is a Lie algebra homomorphism with kernel $\mathfrak{a}$. □

Definition 1.3.8 Commutator Series and Lower Central Series

Let $\mathfrak{g}$ *be a Lie algebra and define a sequence of ideals of* $\mathfrak{g}$ *as follows,*

$$\mathfrak{g}^0=\mathfrak{g}\qquad\mathfrak{g}^1=[\mathfrak{g},\mathfrak{g}]\qquad\mathfrak{g}^k=[\mathfrak{g}^{k-1},\mathfrak{g}^{k-1}]$$

The decreasing sequence $\mathfrak{g}^0\supseteq\mathfrak{g}^1\supseteq\dots$ *is called the commutator series for* $\mathfrak{g}$.

Similarly, the sequence of ideals of $\mathfrak{g}$ *defined by*

$$\mathfrak{g}_0=\mathfrak{g}\qquad\mathfrak{g}_1=[\mathfrak{g},\mathfrak{g}]\qquad\mathfrak{g}_k=[\mathfrak{g},\mathfrak{g}_{k-1}]$$

leads to the decreasing sequence $\mathfrak{g}_0\supseteq\mathfrak{g}_1\supseteq\dots$, *which is called the lower central series for* $\mathfrak{g}$.

$\mathfrak{g}$ is called solvable if $\mathfrak{g}^k=\{0\}$ for some k and nilpotent if $\mathfrak{g}_k=\{0\}$ for some k.

A non-zero solvable Lie algebra $\mathfrak{g}$ has a non-zero abelian ideal, namely the last non-zero $\mathfrak{g}^k$, similarly, a non-zero nilpotent Lie algebra $\mathfrak{g}$ always has non-zero centre equal to the last non-zero $\mathfrak{g}_k$.

For each k, $\mathfrak{g}^k\subseteq\mathfrak{g}_k$ so every nilpotent algebra is solvable.

Theorem 1.3.9 Engel's theorem on nilpotent Lie algebras

Let $V\neq 0$ *be a finite dimensional vector space over a field* $\mathbb{F}$ *and let* $\mathfrak{g}$ *be a Lie algebra of nilpotent endomorphisms of* V. *Then*

(i) $\mathfrak{g}$ *is a nilpotent Lie algebra,*

(ii) *there is some* $v\in V\setminus\{0\}$ *such that* $X(v)=0$ *for all* $X\in\mathfrak{g}$,

(iii) *there is a basis of* V *such that all* $X\in\mathfrak{g}$ *are strict upper triangular matrices with respect to this basis.*

Proof: See [Kn02], pages 46-47.

One key corollary of Engel's theorem is that if $\mathfrak{g}$ is a Lie algebra and $\mathrm{ad}(X)$ is nilpotent for all $X\in\mathfrak{g}$ then $\mathfrak{g}$ is nilpotent.

Proposition 1.3.10 Solvable and nilpotent Lie algebras

Let $\mathfrak{g}$ *be a Lie algebra and let* $\phi : \mathfrak{g} \to \mathfrak{h}$ *be a Lie algebra homomorphism. Then*

(i) *If* $\mathfrak{g}$ *is solvable then any subalgebra of* $\mathfrak{g}$ *is solvable and* $\mathrm{im}(\phi)$ *is also solvable.*

(ii) *If* $\mathfrak{a}$ *is an ideal of* $\mathfrak{g}$ *and both* $\mathfrak{g}/\mathfrak{a}$ *and* $\mathfrak{a}$ *are solvable, then* $\mathfrak{g}$ *is solvable.*

(iii) *Parts (i) and (ii) hold when solvable is replaced by nilpotent.*

(iv) *There is a unique solvable ideal* $\mathfrak{r}$ *of* $\mathfrak{g}$ *containing all solvable ideals of* $\mathfrak{g}$.

Proof: Parts (i) to (iii) are trivial so attention is concentrated on (iv). As the Lie algebra is a finite dimensional vector space it suffices to show that the sum of two solvable Lie algebras is itself solvable.

Therefore suppose $\mathfrak{a}, \mathfrak{b}$ are solvable ideals and let $\mathfrak{h} = \mathfrak{a} + \mathfrak{b}$. As $\mathfrak{a}$ is a solvable ideal in $\mathfrak{h}$ we may take the quotient algebra $\mathfrak{h}/\mathfrak{a} = (\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \cong \mathfrak{b}/(\mathfrak{a} \cap \mathfrak{b})$ by the second isomorphism theorem on rings.
As $\mathfrak{b}, \mathfrak{a} \cap \mathfrak{b}$ are both solvable it follows that $\mathfrak{h}/\mathfrak{a}$ is solvable and hence $\mathfrak{h}$ is solvable. □

The ideal $\mathfrak{r}$ constructed in proposition 1.3.10(iv) is called the radical of $\mathfrak{g}$ and is denoted by $\mathrm{rad}(\mathfrak{g})$.

Definition 1.3.11 Simple and semisimple Lie algebras

A Lie algebra $\mathfrak{g}$ *is called simple if* $\mathfrak{g}$ *is non-abelian and has exactly two ideals, namely* 0 *and* $\mathfrak{g}$. *A Lie algebra* $\mathfrak{g}$ *is called semisimple if it has no nonzero solvable ideals, in other words,* $\mathrm{rad}(\mathfrak{g}) = 0$.

Semisimple Lie algebras are the focus of chapter 2, so it will be useful to state some basic properties of such algebras at this stage.

Proposition 1.3.12 Properties of simple and semisimple Lie algebras

(i) *If* $\mathfrak{g}$ *is a simple Lie algebra then* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

(ii) *Every simple Lie algebra is semisimple.*

(iii) *Every semisimple Lie algebra has centre* 0.

(iv) *If* $\mathfrak{g}$ *is a Lie algebra, then* $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ *is semisimple.*

Proof:

(i) $\mathfrak{g}$ is simple so $[\mathfrak{g}, \mathfrak{g}] = 0$ or $\mathfrak{g}$. If $[\mathfrak{g}, \mathfrak{g}] = 0$ then $\mathfrak{g}$ is abelian which is a contradiction.

(ii) $\mathrm{rad}(\mathfrak{g})$ is an ideal, so $\mathrm{rad}(\mathfrak{g}) = 0$ or $\mathfrak{g}$. If $\mathrm{rad}(\mathfrak{g}) = \mathfrak{g}$ then $\mathfrak{g}$ is solvable and $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$.

(iii) The centre $Z_{\mathfrak{g}}$ is an abelian ideal (and hence solvable), so $0 = \mathrm{rad}(\mathfrak{g}) \supseteq Z_{\mathfrak{g}} = 0$ □

(iv) Let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ be the quotient homomorphism and suppose $\mathfrak{h}$ is a solvable ideal in $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$. Then $\mathfrak{a} = \pi^{-1}(\mathfrak{h})$ is an ideal in $\mathfrak{g}$. $\pi(\mathfrak{a}) = \mathfrak{h}$ is solvable and $\ker \pi \mid_{\mathfrak{a}}$ is solvable as it is contained in $\mathrm{rad}(\mathfrak{g})$. Hence $\mathfrak{a}$ is solvable by 1.3.10(ii) and therefore contained in $\mathrm{rad}(\mathfrak{g})$, so $\mathfrak{h} = \{0\}$. Therefore $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple. □

A Lie group G is said to be simple, semisimple, solvable or nilpotent if and only if its Lie algebra $\mathfrak{g}$ is simple, semisimple, solvable or nilpotent respectively.

Definition 1.3.13 Adjoint representations

Let G *be a Lie group and let* $\mathfrak{g}$ *be its Lie algebra.*

The adjoint representation of G *is the map* $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ *given by*

$$\mathrm{Ad}(g) = d(I_g)_e.$$

The adjoint representation of $\mathfrak{g}$ *is the map* $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ *given by*

$$\mathrm{ad}(X) = (d\,\mathrm{Ad})_e(X).$$

The adjoint representation of $\mathfrak{g}$ has a more usable formulation, given by the following proposition.

Proposition 1.3.14 $\operatorname{ad}(X)(Y) = [X, Y]$

Proof: See [Wa71], proposition 3.47. □

Definition 1.3.15 **The Killing form**

Let $\mathfrak{g}$ be a Lie algebra and pick $X, Y \in \mathfrak{g}$. The Killing form B on $\mathfrak{g}$ is given by $B(X, Y) = \operatorname{Tr}(\operatorname{ad}(X)\operatorname{ad}(Y))$. Moreover, we define $\operatorname{rad}(B) = \{X \in \mathfrak{g} \mid B(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$. If $\mathfrak{g}$ is a Lie algebra and $\operatorname{rad}(B) = \{0\}$ on $\mathfrak{g}$ then the Killing form of $\mathfrak{g}$ is called nondegenerate.

Let $\mathfrak{m} \subseteq \mathfrak{g}$ then $\mathfrak{m}^{\perp} := \{X \in \mathfrak{g} \mid B(X, M) = 0 \text{ for all } M \in \mathfrak{m}\}$.

This definition is meaningful as $\operatorname{ad}(X)\operatorname{ad}(Y)$ is a linear transformation on $\mathfrak{g}$ and is therefore represented by a $\dim(\mathfrak{g}) \times \dim(\mathfrak{g})$ matrix and its trace can be calculated.

Proposition 1.3.16 **Properties of the Killing form**

Let $\mathfrak{g}$ be a Lie algebra.

(i) *The Killing form is a symmetric bilinear form on $\mathfrak{g}$.*

(ii) *For all X, Y, Z in $\mathfrak{g}$, $B(X, [Y, Z]) = B([X, Y], Z)$.*

(iii) *If $a \in \operatorname{Aut}(\mathfrak{g})$, then $B(aX, aY) = B(X, Y)$.*

Proof: The first part follows because the adjoint map is linear and the trace function is symmetric and linear. For the second part, expand both sides and use the symmetry of the trace to obtain the desired result.

Finally we note that $\operatorname{ad}(aX) = a(\operatorname{ad}(X))a^{-1}$, as

$$\operatorname{ad}(aX)Y = [aX, Y] = a[X, a^{-1}Y] = (a(\operatorname{ad}(X))a^{-1})(Y).$$

Therefore,

$$\begin{aligned} B(aX, aY) &= \operatorname{Tr}(\operatorname{ad}(aX)\operatorname{ad}(aY)) \\ &= \operatorname{Tr}(a(\operatorname{ad}(X))a^{-1}a(\operatorname{ad}(Y))a^{-1}) \\ &= \operatorname{Tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) \\ &= B(X, Y). \end{aligned}$$ □

The result of part (ii) can also be written in the form $B([X, Y], Z) = -B(Y, [X, Z])$.

1.4 Characterisation of semisimple Lie algebras

Chapters 2 and 3 are primarily concerned with semisimple Lie algebras and groups. In this section we provide a sufficient condition for a finite dimensional Lie algebra to be semisimple and use this to characterise all semisimple Lie algebras in terms of simple Lie algebras.

Theorem 1.4.1 **Cartan's criterion for solvability**

A Lie algebra $\mathfrak{g}$ is solvable if and only if $B(X, Y) = 0$ for all $X \in \mathfrak{g}$ and all $Y \in [\mathfrak{g}, \mathfrak{g}]$.

The proof of theorem 1.4.1 can be found in [Kn02] on pages 51-54.

To prove the following corollary we need an intermediate result, namely that for a Lie algebra $\mathfrak{g}$, $\operatorname{rad}(B) \subseteq \mathfrak{r} = \operatorname{rad}(\mathfrak{g})$.

It suffices to show that $\operatorname{rad}(B)$ is a solvable ideal. Let $H \in \operatorname{rad}(B)$ and let $X_1, X_2 \in \mathfrak{g}$. Then

$$B([X_1, H], X_2) = -B(H, [X_1, X_2])$$

and as a result of this $[X_1, H] \in \text{rad}(B)$ and therefore $\text{rad}(B)$ is an ideal of $\mathfrak{g}$.

Pick $\mathfrak{s}$ such that $\mathfrak{g} = \text{rad}(B) \oplus \mathfrak{s}$. This is possible as any subspace has a basis which can be extended to a basis of the whole space, $\mathfrak{s}$ is just the span of the vectors added by this process. If $X \in \text{rad}(B)$, then in terms of the basis discussed, the adjoint map extends the zero map over $\mathfrak{s}$, by proposition 1.3.14. Thus the trace of $\text{ad}(X)$ is the same as if it were restricted to the ideal $\text{rad}(B)$.

Hence, if C is the Killing form on $\text{rad}(B)$ then $C(X, Y) = \text{Tr}(\text{ad}(X)\text{ad}(Y)) = B(X, Y) = 0$ and so by theorem 1.4.1, $\text{rad}(B)$ is solvable. □

Corollary 1.4.2 Cartan's criterion for semisimplicity

A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form on $\mathfrak{g}$ is nondegenerate.

Proof: If B is degenerate then $\text{rad}(B) \neq 0$ and therefore $\text{rad}(\mathfrak{g}) \neq 0$ by the previous result, so $\mathfrak{g}$ is not semisimple.

Conversely, if $\mathfrak{g}$ is not semisimple then $\text{rad}(\mathfrak{g}) \neq 0$. However, $\text{rad}(\mathfrak{g})$ is solvable so there is an integer l such that $(\text{rad}(\mathfrak{g}))^l = 0$ and $(\text{rad}(\mathfrak{g}))^{l-1} = \mathfrak{a}$ is a nonzero abelian ideal in $\mathfrak{g}$.

Set $\mathfrak{s}$ so that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{s}$ as a vector space. If $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$ then $\text{ad}(X)$ is zero on $\mathfrak{a}$ as $\mathfrak{a}$ is an abelian ideal. Also the adjoint map extends the zero map over $\mathfrak{s}$ (as in the preceding comments). Hence, $\text{Tr}(\text{ad}(X)\text{ad}(Y)) = 0$ and $\mathfrak{a} \subseteq \text{rad}(B)$, so B is degenerate. □

Theorem 1.4.3 Jordan decomposition

Let $\mathbb{F}$ be an algebraically closed field and let V be a finite dimensional vector space over $\mathbb{F}$. Then each $\psi \in \text{End}_{\mathbb{F}}(V)$ can be uniquely decomposed as $\psi = s + n$ where s is diagonalisable, n is nilpotent and $sn = ns$.

Proof: See [Hu78], section 4.2. □

Theorem 1.4.4 Characterisation of semisimple Lie algebras

A Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} = \bigoplus_{i=1}^m \mathfrak{g}_j$ where each $\mathfrak{g}_j$ is a simple ideal of $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple then this decomposition is unique and the only ideals of $\mathfrak{g}$ are the sums of the $\mathfrak{g}_j$.

Proof: Suppose $\mathfrak{g} = \bigoplus_{j=1}^m \mathfrak{g}_j$ where each $\mathfrak{g}_j$ is a simple ideal of $\mathfrak{g}$. Let P_i be the projection of $\mathfrak{g}$ onto $\mathfrak{g}_i$ and let $\mathfrak{a}$ be any ideal in $\mathfrak{g}$ and set $\mathfrak{a}_i = P_i(\mathfrak{a})$. $\mathfrak{a}_i$ is an ideal in $\mathfrak{g}_i$ as

$$[P_i(A), X_i] = P_i([A, X_i]) \in P_i(\mathfrak{a}) = \mathfrak{a}_i \qquad \text{for all } A \in \mathfrak{a}.$$

$\mathfrak{g}_i$ is simple so $\mathfrak{a}_i = 0$ or $\mathfrak{a}_i = \mathfrak{g}_i$. If $\mathfrak{a}_i = \mathfrak{g}_i$, then

$$\mathfrak{g}_i = [\mathfrak{g}_i, \mathfrak{g}_i] = [\mathfrak{g}_i, \mathfrak{a}_\mathfrak{i}] \subseteq [\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}.$$

As $\mathfrak{g} = \bigoplus_{j=1}^m \mathfrak{g}_j$, it follows that $\mathfrak{a} = \mathfrak{a} \cap \mathfrak{g} = \bigoplus_{i=1}^m (\mathfrak{a} \cap \mathfrak{g}_i) = \bigoplus_{\mathfrak{g}_i \subseteq \mathfrak{a}} \mathfrak{g}_i$.

This proves the uniqueness of the decomposition and the structure of the ideals. Also

$$\left[\bigoplus_{\mathfrak{g}_i \subseteq \mathfrak{a}} \mathfrak{g}_i, \bigoplus_{\mathfrak{g}_j \subseteq \mathfrak{a}} \mathfrak{g}_j\right] = \bigoplus_{\mathfrak{g}_i \subseteq \mathfrak{a}} [\mathfrak{g}_i, \mathfrak{g}_i] = \bigoplus_{\mathfrak{g}_i \subseteq \mathfrak{a}} \mathfrak{g}_i = \mathfrak{a}.$$

Therefore $\mathfrak{a}$ is solvable if and only if $\mathfrak{a} = 0$. Thus $\text{rad}(\mathfrak{g}) = 0$, so by Cartan's criterion for semisimplicity, $\mathfrak{g}$ is semisimple.

Suppose $\mathfrak{g}$ is semisimple. Let $\mathfrak{a}$ be a minimal (by inclusion) non-zero ideal of $\mathfrak{g}$.

Write $\mathfrak{a}^\perp = \{X \in \mathfrak{g} \,|\, B(X, A) = 0 \text{ for all } A \in \mathfrak{a}\}$. $\mathfrak{a}^\perp$ is an ideal, as

$$B([X, H], A) = B(H, -[X, A]) \in B(H, \mathfrak{a}) = 0 \text{ for all } A \in \mathfrak{a},\ X \in \mathfrak{g} \text{ and } H \in \mathfrak{a}^\perp.$$

Therefore $\text{rad}(B|_{\mathfrak{a} \times \mathfrak{a}}) = \mathfrak{a} \cap \mathfrak{a}^\perp = 0$ or $\mathfrak{a}$, by the minimality of $\mathfrak{a}$.

Suppose $\text{rad}(B|_{\mathfrak{a}\times\mathfrak{a}}) = \mathfrak{a}$, then $B(A_1, A_2) = 0$ for all $A_1, A_2 \in \mathfrak{a}$. $\mathfrak{a}$ is an ideal, so $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$. By theorem 1.4.1, $\mathfrak{a}$ is solvable, contradicting the semisimplicity of $\mathfrak{g}$.

Thus $\mathfrak{a} \cap \mathfrak{a}^{\perp} = 0$. The Killing form is non-degenerate on $\mathfrak{a}$, so $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ as a vector space and both $\mathfrak{a}$ and $\mathfrak{a}^{\perp}$ are ideals of $\mathfrak{g}$. Also, the non-degeneracy shows $\mathfrak{a}$ is nonabelian.

Suppose $\mathfrak{b} \subseteq \mathfrak{a}$, then $[\mathfrak{b}, \mathfrak{a}^{\perp}] = 0$, so $\mathfrak{b}$ is an ideal of $\mathfrak{g}$. Moreover, $\mathfrak{b} = 0$ or $\mathfrak{a}$ by the minimality of $\mathfrak{a}$. Thus $\mathfrak{a}$ is simple.

Finally, note that any ideal of $\mathfrak{a}^{\perp}$ is an ideal of $\mathfrak{g}$ and therefore $\text{rad}(\mathfrak{a}^{\perp}) = 0$ so $\mathfrak{a}^{\perp}$ is semisimple, by corollary 1.4.2. We repeat the argument with $\mathfrak{a}^{\perp}$ and complete the proof by induction on $\dim(\mathfrak{g})$. ☐

Proposition 1.4.5 *The Lie algebra of* $\text{Aut}(\mathfrak{g})$ *is* $\text{Der}(\mathfrak{g})$*, the space of all derivations on* $\mathfrak{g}$*. If* $\mathfrak{g}$ *is semisimple then* $\text{Der}(\mathfrak{g}) = \text{ad}(\mathfrak{g})$.

Proof: This is a special case of [Wa71], theorem 3.54. ☐

Later, we will need a way of changing the base field of a Lie algebra between $\mathbb{R}$ and $\mathbb{C}$. With this goal in mind, we make the following definition.

Definition 1.4.6 Complexifications and real forms

Given a real Lie algebra $\mathfrak{g}$ *we can define a tensor product* $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ *as follows.*
Let $\{X_1, \ldots, X_n\}$ *be a basis for* $\mathfrak{g}$ *over* $\mathbb{R}$ *and define* $X \otimes c = \sum_{i=1}^{n} ca_iX_i$*, where* $X = \sum_{i=1}^{n} a_iX_i$*.*
To make $\mathfrak{g}^{\mathbb{C}}$ *into a Lie algebra define* $[X \otimes c, Y \otimes d] = [X, Y] \otimes cd$*.*

$\mathfrak{g}^{\mathbb{C}}$ *is called the complexification of* $\mathfrak{g}$*.*

Note that $\{X_1, \ldots, X_n\}$ is a basis for $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{C}$ and $\{X_1, \ldots, X_n, iX_1, \ldots, iX_n\}$ is a basis for $\mathfrak{g}^{\mathbb{C}}$ over $\mathbb{R}$. So $\dim_{\mathbb{C}}(\mathfrak{g}^{\mathbb{C}}) = \dim_{\mathbb{R}}(\mathfrak{g})$ and $\dim_{\mathbb{R}}(\mathfrak{g}^{\mathbb{C}}) = 2\dim_{\mathbb{R}}(\mathfrak{g})$

Conversely, given a complex Lie algebra $\mathfrak{g}$ we define $\mathfrak{g}^{\mathbb{R}}$ to be the vector space $\mathfrak{g}$ considered over the field $\mathbb{R}$ instead of $\mathbb{C}$.

The key result which is obtained from this new basis is $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}} = \mathfrak{g} \oplus i\mathfrak{g}$. Consequently, given a complex Lie algebra $\mathfrak{g}$, any real Lie algebra $\mathfrak{h}$ which satisfies $\mathfrak{g}^{\mathbb{R}} = \mathfrak{h} \oplus i\mathfrak{h}$ is called a real form of $\mathfrak{g}$. As should be expected, any real Lie algebra is a real form of its complexification.

1.5 Connected Lie groups

This section contains many of the most important and fundamental results relating Lie algebras and Lie groups. Many of them have involved proofs which are omitted. We will make use of these results (often without directly referring to them) throughout the remaining chapters.

Theorem 1.5.1 Equality of Lie group homomorphisms

Let G *be a connected Lie group and let* $\psi,\ \phi : G \to H$ *be two Lie group homomorphisms whose differentials* $d\psi,\ d\phi : \mathfrak{g} \to \mathfrak{h}$ *are identical. Then* $\psi = \phi$*.*

Proof: See [Wa71], theorem 3.16. ☐

Proposition 1.5.2 Generating connected Lie groups

Let G *be a connected Lie group and let* U *be an open neighbourhood of* 1_G*. Then* $G = \bigcup_{n=1}^{\infty} U^n$*, where* U^n *is the set of all* n*-fold products of elements of* U*.*

Proof: Set $V = (U \cap U^{-1})$. The set U^{-1} is also an open neighbourhood of 1_G as the inversion map is continuous. Therefore V is an open neighbourhood of 1_G. Define

$$H = \bigcup_{n=1}^{\infty} V^n \subseteq \bigcup_{n=1}^{\infty} U^n.$$

H is a subgroup of G and is open in G as each element $h \in H$ lies in the open neighbourhood $hV \subseteq H$. Thus all cosets of H in G are open, in particular, $G \setminus H$ is open, so H is a closed subgroup of G. As G is connected and $H \neq \{1_G\}$, it follows that $H = G$. ☐

Theorem 1.5.3 Existence and uniqueness of connected Lie subgroups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. There is a unique connected Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$.

Proof: See [Wa71], theorem 3.19. □

Corollary 1.5.4 Subgroup-subalgebra correspondence

Let G be a Lie group with Lie algebra $\mathfrak{g}$. There is a 1-1 correspondence between the connected Lie subgroups of G and the subalgebras of $\mathfrak{g}$. Moreover, the correspondence maps normal subgroups to ideals.

Proof: This follows immediately from theorem 1.5.3.

We will use this corollary regularly. At this time we use it to define the connected Lie subgroup $\mathrm{Int}(\mathfrak{g})$ to be the unique connected subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$. If $\mathfrak{g}$ is semisimple then by proposition 1.4.5, $\mathrm{Int}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})_0$.

Theorem 1.5.5 Uniqueness of Lie group homomorphisms

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If G is simply connected then for each Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{h}$ there exists a unique Lie group homomorphism $\phi : G \to H$ with $d\phi = \psi$.

Proof: See [Wa71], theorem 3.27.

Theorem 1.5.6 Campbell-Baker-Hausdorff formula

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. The element $Z \in \mathfrak{g}$ satisfying the equation $\exp(Z) = \exp(X)\exp(Y)$ *is given by the formula*

$$Z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{1 \le i \le n} (r_1! s_1! \dots r_n! s_n!)^{-1} \sum_{r_i+s_i=1}^{\infty} \sum_{i=1}^{n} (r_i + s_i)^{-1} [X^{r_1} Y^{s_1} \dots X^{r_n} X^{s_n}],$$

where the notation $[X^{r_1} Y^{s_1} \dots X^{r_n} X^{s_n}]$ *is a shorthand for the iterated bracket*

$$[\underbrace{X, [X, \dots [X,}_{r_1} \underbrace{[Y, [Y, \dots [Y,}_{s_1} [X, \dots [\underbrace{X, [X, \dots [X,}_{r_n} \underbrace{[Y, [Y, \dots, Y}_{s_n}]]] \dots]]$$

We note first that this iterated bracket is 0 if $s_n > 1$ or $s_n = 0$ and $r_n > 1$. The first three terms of this expression are given by

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] - [Y, [X, Y]]) + \dots \ .$$

A proof of this result can be found in [Kn02] on pages 669-682.

1.6 Representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

In this section we concentrate on the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. The set $\{e, f, h\}$ is a basis for the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notice that $[e, f] = h, [h, e] = 2e$ and $[h, f] = -2f$ and hence $[\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C})] = \mathfrak{sl}_2(\mathbb{C})$.

Definition 1.6.1 *Let $\psi : \mathfrak{g} \to gl(V)$ be a representation of $\mathfrak{sl}_2(\mathbb{C})$ on a finite dimensional complex vector space V.*

(i) *An invariant subspace under ψ is a subspace $U \subseteq V$ such that $\psi(X)U \subseteq U$ for all $X \in \mathfrak{sl}_2(\mathbb{C})$.*

(ii) *ψ is called irreducible if the only invariant subspaces are 0 and V.*

(iii) *If* $\psi' : \mathfrak{sl}_2(\mathbb{C}) \to GL(V')$ *is a representation of* $\mathfrak{sl}_2(\mathbb{C})$ *on a finite dimensional complex vector space* V', *then* ψ *and* ψ' *are said to be equivalent if there is an isomorphism* $E : V \to V'$ *such that* $E\psi(X) = \psi'(X)E$ *for all* $X \in \mathfrak{sl}_2(\mathbb{C})$.

Theorem 1.6.2 Irreducible representations of $\mathfrak{sl}_2(\mathbb{C})$

For each $m \in \mathbb{N}$ *there is a unique (up to equivalence) irreducible complex-linear representation* $\pi : \mathfrak{sl}_2(\mathbb{C}) \to GL_m(\mathbb{C})$. *Moreover, there is a basis* $\{v_0, \dots v_{m-1}\}$ *for* V *such that*

(i) $\pi(h)v_i = (m - 1 - 2i)v_i$,

(ii) $\pi(e)v_0 = 0$,

(iii) $\pi(f)v_i = v_{i+1}$, *where* $v_m := 0$,

(iv) $\pi(e)v_i = i(m - i)v_{i-1}$, *for* $1 \leq i \leq m - 1$.

Proof: We prove that properties (i)-(iv) define an irreducible complex-linear representation as claimed. The proof of uniqueness is omitted and can be found in [Kn02] on page 63.

Define $\pi(h), \pi(e), \pi(f)$ as given by (i) - (iv) and extend linearly to define π on $\mathfrak{sl}_2(\mathbb{C})$. Simple calculations show that $\pi([h, e]) = [\pi(h), \pi(e)]$, $\pi([h, f]) = [\pi(h), \pi(f)]$ and $\pi([e, f]) = [\pi(e), \pi(f)]$ so π is a complex-linear representation.

To prove the representation is irreducible, suppose U is a nontrivial subspace of V, hence $v_i \in U$ for some i. By (iv) we can apply $\pi(e)$ to v_i to show $v_0, \dots, v_{i-1} \in U$ and by (iii) we can apply $\pi(f)$ to v_i to show $v_{i+1}, \dots, v_{m-1} \in U$, so $U = V$. □

Later results will rely heavily on the uniqueness of such representations.

Chapter 2

Complex semisimple Lie algebras

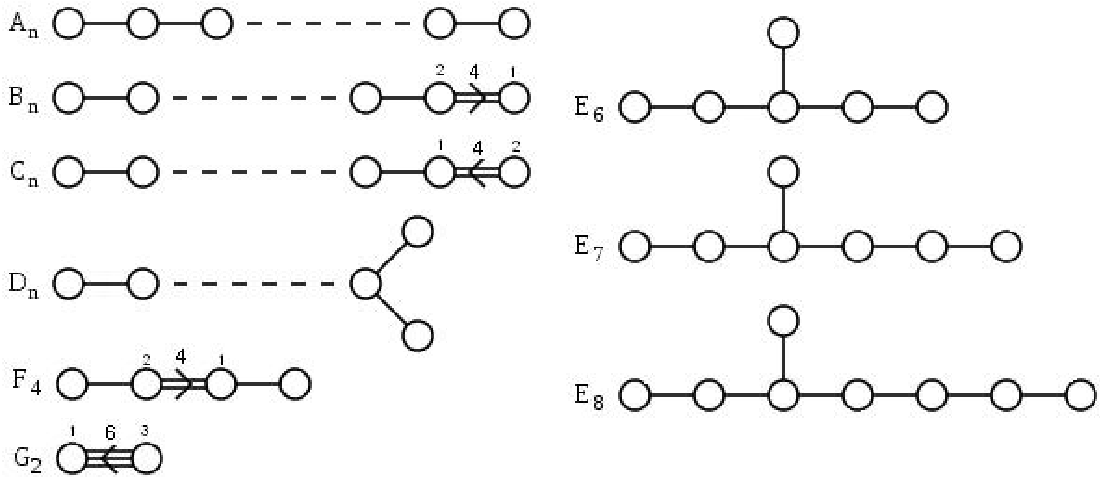
The purpose of this chapter is to introduce complex semisimple Lie algebras and classify them up to isomorphism. This is achieved by relating such Lie algebras bijectively to a collection of diagrams called Dynkin diagrams. These diagrams are then classified using Cartan matrices. The main reference for this chapter is [Kn02] chapters 1 and 2.

2.1 The classification theorem

Theorem 2.1.1 Classification theorem

To each complex semisimple Lie algebra we can associate a Dynkin diagram in which every connected component is isomorphic to exactly one such diagram from figure 2.1.2*. Moreover, if* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are two complex semisimple Lie algebras, then* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *are isomorphic if and only if they have isomorphic Dynkin diagrams.*

Figure 2.1.2 Dynkin diagrams



2.2 Example: $\mathfrak{sl}_{n+1}(\mathbb{C})$, for $n \geq 1$

Restricting the classification theorem just to this case we obtain the following theorem.

Theorem 2.2.1 $\mathfrak{g} := \mathfrak{sl}_{n+1}(\mathbb{C}) = \{A \in M_{n+1}(\mathbb{C}) \mid \mathrm{Tr}(A) = 0\}$ *is the unique (up to isomorphism) complex simple Lie algebra with Dynkin diagram* A_n.

We will make no attempt to prove the uniqueness in this section.

The first task is to show that $\mathfrak{g}$ is actually a complex Lie algebra. This result was established by showing that $\mathfrak{g}$ is the Lie algebra of $SL_{n+1}(\mathbb{C})$, in this section we will prove the result directly.

Note that $\mathfrak{g}$ is certainly a complex vector space and a ring with respect to the multiplication

$[A,B] = AB - BA$ since $\mathrm{Tr}(AB - BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(BA) = \mathrm{Tr}(AB) - \mathrm{Tr}(AB) = 0$.

It remains to show that this bracket is antisymmetric and satisfies the Jacobi identity. The first is obvious from the definition, and

$$\begin{aligned}[A,[B,C]] + [B,[C,A]] + [C,[A,B]] &= A(BC - CB) - (BC - CB)A + B(CA - AC) \\ &\quad -(CA - AC)B + C(AB - BA) - (AB - BA)C \\ &= 0\end{aligned}$$

We will define the following elements of $\mathfrak{g}$.

For $i \neq j$ we define $E_{i,j}$ to be zero except in the $(i,j)^{\text{th}}$ entry where it takes value 1 and we define $H_{i,j}$ to be zero except in the i^{th} diagonal entry where it takes value 1 and the j^{th} diagonal entry where it takes value -1.

Notice that $\mathcal{B} := \{E_{i,j} \mid i \neq j, 1 \leq i,j \leq n+1\} \cup \{H_{i,i+1} \mid 1 \leq i \leq n\}$ is a basis for $\mathfrak{g}$ as a vector space.

It is easy to verify that $\mathfrak{g}$ is semisimple using Cartan's criterion for semisimplicity, as the Killing form is non-degenerate on $\mathfrak{g}$. Later we will use results from this section to show that $\mathfrak{g}$ is actually a simple Lie algebra.

Set $\mathfrak{h}$ to be the Lie subalgebra of $\mathfrak{g}$ with basis $\{H_{i,i+1} \mid 1 \leq i \leq n\}$, this subalgebra is abelian and thus has the key property that $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h}) = \{\mathrm{ad}(H) : \mathfrak{g} \to \mathfrak{g} \mid H \in \mathfrak{h}\}$ is simultaneously diagonalisable with respect to the basis $\mathcal{B}$.

Moreover, if we define a collection of linear functionals $e_i : \mathfrak{h} \to \mathbb{C}$, where $e_i(H)$ is the i^{th} diagonal entry of H, then we see that given two distinct indices i and j,

$$(\mathrm{ad}(H))E_{i,j} = [H, E_{i,j}] = e_i(H)E_{i,j} - e_j(H)E_{i,j} = (e_i - e_j)(H)E_{i,j}.$$

We can think of $E_{i,j}$ as a simultaneous eigenvector of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ whose eigenvalue is the linear functional $e_i - e_j \in \mathfrak{h}^*$.

Using these simultaneous eigenvectors we can decompose $\mathfrak{g}$ as follows.

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathfrak{g}_{e_i - e_j} \quad \text{where} \quad \mathfrak{g}_{e_i - e_j} := \{X \in \mathfrak{g} \mid \mathrm{ad}(H)(X) = (e_i - e_j)(H)(X)\}.$$

Similarly, we could define $\mathfrak{g}_0 := \{X \in \mathfrak{g} \mid \mathrm{ad}(H)(X) = 0\}$ and it is clear that $\mathfrak{g}_0 = \mathfrak{h}$.

The decomposition of $\mathfrak{g}$ above is called the root-space decomposition, $\mathfrak{h}$ is called a Cartan subalgebra and the linear functionals $e_i - e_j$ for $i \neq j$ are called roots, the set of roots will be denoted by Φ. For each root $e_i - e_j$ we will call $E_{i,j}$ the root vector associated to the root $e_i - e_j$. The set of roots span $\mathfrak{h}^*$, as the set of e_i span $\mathfrak{h}^*$.

We define $\mathfrak{h}_0$ to the real form of $\mathfrak{h}$ consisting of those diagonal matrices with real entries. The linear functionals e_i (considered now as elements of $(\mathfrak{h}_0)^*$) span $(\mathfrak{h}_0)^*$ and their sum is the trace function on $\mathfrak{h}_0$ which is the zero functional as $\mathfrak{h}_0 \subseteq \mathfrak{g}$. There are $n+1$ vectors e_i (which span $(\mathfrak{h}_0)^*$) and $(\mathfrak{h}_0)^*$ has dimension n so it suffices to impose one suitable restriction on the coefficients in the sum $\sum_{j=1}^{n+1} a_j e_j$ to obtain a unique expression for each functional. Such a condition is given by $\sum_{j=1}^{n+1} a_j = 0$.

A non-zero functional $\phi = \sum_{j=1}^{n+1} a_j e_j$ (with $\sum_{j=1}^{n+1} a_j = 0$) is said to be positive if and only if the first non-zero a_j is positive. If ϕ is positive then we say that $-\phi$ is negative. Notice that a sum of positive elements is positive and a sum of negative elements is negative.

With respect to this ordering the positive elements of Φ behave as follows:

$$\begin{aligned}e_1 - e_{n+1} &> e_1 - e_n > \cdots > e_1 - e_2 \\ &> e_2 - e_{n+1} > e_2 - e_n > \cdots > e_2 - e_3 \\ &> \cdots > e_{n-1} - e_{n+1} > e_{n-1} - e_n > e_n - e_{n+1} > 0.\end{aligned}$$

We say that a positive root is simple if it cannot be expressed as the some of at least two other positive roots. The set of simple roots of $\mathfrak{g}$ is $\Pi = \{e_i - e_{i+1} \,|\, 1 \leq i \leq n\}$

Although this step is not essential for constructing the Dynkin diagram of $\mathfrak{g}$, it is very informative and will be a crucial ingredient when proving theorem 2.1.1 and in later chapters (for example in definition 7.1.7).

Consider the subspace $\mathfrak{sl}_{i,j}$ of $\mathfrak{g}$ with basis $\{E_{i,j}, E_{j,i}, H_{i,j}\}$ for some $1 \leq i < j \leq n+1$. Then the map

$$\psi : \mathfrak{sl}_{i,j} \longrightarrow \mathfrak{sl}_2(\mathbb{C}) \text{ given by}$$

$$\psi(E_{i,j}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \psi(E_{j,i}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \psi(H_{i,j}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a Lie algebra isomorphism with respect to the Lie brackets of $\mathfrak{g}$ and $\mathfrak{sl}_2(\mathbb{C})$. As this holds whenever $i < j$ we can see that $\mathfrak{g}$ is spanned by copies of $\mathfrak{sl}_2(\mathbb{C})$.

In order to obtain the Dynkin diagram we require a bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}^*$. This is given by taking the $\mathbb{C}$-linear extension of

$$\langle e_i - e_j, e_k - e_l \rangle = B(H_{i,j}, H_{k,l}) = (e_i - e_j)(H_{k,l}) = (e_k - e_l)(H_{i,j}),$$

where B is the Killing form of $\mathfrak{g}$. This defines a bilinear form on $\mathfrak{h}^*$ as Φ spans $\mathfrak{h}^*$ and B is a symmetric bilinear form.

With respect to this inner product $\langle e_i - e_j, e_i - e_j \rangle = 2$ whenever $i \neq j$.

Let $e_i - e_j \in \Phi$ (where $e_i - e_j$ is restricted to the space $(\mathfrak{h}_0)^*$). We define a root reflection

$$s_{i,j}(\psi) = \psi - \frac{2\langle \psi, e_i - e_j \rangle}{\langle e_i - e_j, e_i - e_j \rangle}(e_i - e_j).$$

For each root this is a reflection in the hyperplane orthogonal to $\mathbb{R}(e_i - e_j)$ which permutes Φ.

The Weyl group W is the finite subgroup of the orthogonal group generated by the set of reflections $S := \{s_a \,|\, a \in \Pi\}$.

Now that we have the collection of root reflections in place we may define the Cartan matrix of Φ. This $n \times n$ matrix $A = (a_{ij})$ is given by

$$a_{ij} = \frac{2\langle e_i - e_{i+1}, e_j - e_{j+1} \rangle}{\langle e_i - e_{i+1}, e_i - e_{i+1} \rangle}.$$

We note immediately that $a_{ii} = 2$ for all i. It also follows that $a_{ij} = -1$ whenever $|i - j| = 1$ and $a_{ij} = 0$ otherwise.

So in the case $n = 3$, the Cartan matrix of $\mathfrak{sl}_4(\mathbb{C})$ is $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Finally we are in a position to define the Dynkin diagram of $\mathfrak{g}$. The Dynkin diagram is the graph with vertex set Π and edges between two vertices $e_i - e_{i+1}$ and $e_j - e_{j+1}$ if and only if $a_{ij}a_{ji} \neq 0$.

Every edge is assigned a label equal to 3 and each vertex $e_i - e_{i+1}$ is given a label equal to $c\langle e_i - e_{i+1}, e_i - e_{i+1} \rangle$ where $c > 0$ is some constant which does not depend on i, chosen minimally so that all labels take integer values. In this case $\langle e_i - e_{i+1}, e_i - e_{i+1} \rangle = 2$ for all i so we choose $c = \frac{1}{2}$.

We can see from the diagram 2.1.2 that every vertex is not labelled. It is conventional that labels are only written on adjacent vertices which have distinct labels. Often an edge connecting vertices with distinct labels is displayed as a directed edge towards the vertex of smaller label.

Thus $\mathfrak{g}$ has Dynkin diagram A_n. In the chapter on buildings we will see that Dynkin diagrams are in fact examples of Coxeter diagrams, where the group associated to the Coxeter diagram is, in fact, the Weyl group of root reflections. This will enable us to prove that $\mathfrak{g}$ has Weyl group S_{n+1}.

In this section, many things have been assumed, most commonly that the choices made (Cartan subalgebra, ordering of Φ, choice of simple roots Π) have no impact on the resulting Cartan matrix, Dynkin diagram and Weyl group. The real triumph of the classification theorem is in proving these assumptions are justified. As with this section the place to start is with Cartan subalgebras and roots.

2.3 Cartan subalgebras and roots

This section abstracts the constructions from the previous section, introducing Cartan subalgebras and root systems. We will then show how each pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ gives rise to an abstract root system.

Definition 2.3.1 Cartan subalgebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a Cartan subalgebra if it is maximal by inclusion amongst the abelian Lie subalgebras $\mathfrak{a}$ of $\mathfrak{g}$.

If a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is abelian, it follows that any two linear transformations from $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$ commute. To see this, notice that for any $X \in \mathfrak{g}$,

$$\begin{aligned}(\mathrm{ad}(A)\mathrm{ad}(A'))(X) &= [A,[A',X]] = -[A',[X,A]] - [X,[A,A']] && \text{(by the Jacobi identity)} \\ &= -[A',[X,A]] && \text{(since } \mathfrak{a} \text{ is abelian)} \\ &= [A',[A,X]] = (\mathrm{ad}(A')\mathrm{ad}(A))(X).\end{aligned}$$

We know from basic representation theory that any finite set of commuting linear transformations is simultaneously diagonalisable and therefore the underlying space has a basis of simultaneous eigenvectors whose eigenvalues are constant. However, $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$ is certainly not finite. Instead we use the fact that $\mathfrak{a}$ is finite dimensional, with basis $\{H_1, \ldots, H_k\}$ say. Thus, under the linear transformations $\{\mathrm{ad}(H_1), \ldots, \mathrm{ad}(H_k)\}$, $\mathfrak{g}$ has a basis of simultaneous eigenvectors whose eigenvalues are constant.

Suppose $X \in \mathfrak{g}$ lies in one of these simultaneous eigenspaces, so for each i there is some constant c_i such that $[H_i, X] = c_i X$.

Let $H \in \mathfrak{a}$, then $H = \sum_{i=1}^{k} a_i H_i$ and therefore

$$\begin{aligned}[H,X] &= \textstyle\sum_{i=1}^{k} a_i [H_i, X_i] \\ &= \left(\textstyle\sum_{i=1}^{k} a_i c_i\right) X\end{aligned}$$

so X lies in a simultaneous eigenspace under the set of transformations $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{a})$ with eigenvalue $\sum_{i=1}^{k} a_i e_i$, which is a linear functional on $\mathfrak{a}$.

The same construction applies equally well to infinite dimensional Lie algebras providing every maximal (by inclusion) abelian subalgebra is finite dimensional.

Proposition 2.3.2 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then*

$$\mathfrak{g} = \bigoplus_{a \in \mathfrak{h}^*} \mathfrak{g}_a \quad \text{where} \quad \mathfrak{g}_a = \{X \in \mathfrak{g} \,|\, [H,X] = a(H)X \;\text{ for all }\; H \in \mathfrak{h}\}.$$

Furthermore,$\mathfrak{g}_0 = \mathfrak{h}$ and $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$, for all $a, b \in \mathfrak{h}^*$.

Proof: The fact that all $X \in \mathfrak{g}$ lie in some $\mathfrak{g}_a$ has already been covered.

Suppose that $X \in \mathfrak{g}_a \cap \mathfrak{g}_b$. Then $a(H)X = [H,X] = b(H)X$ for all $H \in \mathfrak{h}$,
so $a = b$.

Let $X \in \mathfrak{g}_a$, $Y \in \mathfrak{g}_b$ and $H \in \mathfrak{h}$. Then

$$\begin{aligned}(\mathrm{ad}(H) - (a(H)+b(H))I)[X,Y] &= [H,[X,Y]] - a(H)[X,Y] - b(H)[X,Y] \\ &= [(\mathrm{ad}(H) - a(H)I)X, Y] + [X, (\mathrm{ad}(H) - b(H)I)Y] \\ &= 0 \quad \text{as } X \in \mathfrak{g}_a \text{ and } Y \in \mathfrak{g}_b\end{aligned}$$

$\mathfrak{g}_0$ is precisely the subspace of $\mathfrak{g}$ consisting of all elements of $\mathfrak{g}$ which commutes with $\mathfrak{h}$, as $\mathfrak{h}$ was chosen maximally it follows that $\mathfrak{g}_0 = \mathfrak{h}$. □

The decomposition given by proposition 2.3.2 is called the root-space decomposition of $\mathfrak{g}$. A linear functional $a \in \mathfrak{h}^* \setminus \{0\}$ is called a root if and only if $\mathfrak{g}_a \neq \{0\}$. The set of roots will be denoted by $\Phi(\mathfrak{g}, \mathfrak{h})$ to show its dependence on the choice of $\mathfrak{h}$, when this is clear from the context we will simply write Φ. Nonzero members of $\mathfrak{g}_a$ are called root vectors for the root a.
As was mentioned at the start of this section the next important result will be to prove that the set of roots Φ given by the root-space decomposition forms an abstract root system. With this in mind we give the following definition and theorem.

Definition 2.3.3 Abstract Root Systems

Let $(V, \langle\cdot,\cdot\rangle)$ be finite dimensional real vector space equipped with a bilinear form. An abstract root system in V is a finite subset $\Phi \subseteq V \setminus \{0\}$ with the following properties:

(i) *Φ spans V,*

(ii) *if $a, b \in \Phi$ then $s_a(b) = b - \frac{2\langle b,a\rangle}{\langle a,a\rangle} a \in \Phi$,*

(iii) *if $a, b \in \Phi$ then $\frac{2\langle b,a\rangle}{\langle a,a\rangle} \in \mathbb{Z}$.*

An abstract root system is said to be reduced if $a \in \Phi$ implies $2a \notin \Phi$.

Two abstract root systems (V, Φ) and (V', Φ') are isomorphic is there is a vector space isomorphism $\psi : V \to V'$ with $\psi(\Phi) = \Phi'$.

Theorem 2.3.4 *The root system of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ forms a reduced abstract root system in $(\mathfrak{h}_0)^*$.*

The remainder of this section will essentially be a proof of theorem 2.3.4.

Proposition 2.3.5 Basic properties of root spaces

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$.

(i) *If $a, b \in \Phi \cup \{0\}$ and $a + b \neq 0$, then $B(\mathfrak{g}_a, \mathfrak{g}_b) = 0$.*

(ii) *If $a \in \Phi \cup \{0\}$ then B is non-singular on $\mathfrak{g}_a \times \mathfrak{g}_{-a}$.*

(iii) *If $a \in \Phi$ then $-a \in \Phi$.*

(iv) *$B|_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate.*

(v) *Φ spans $\mathfrak{h}^*$.*

Proof:

(i) $\mathrm{ad}(\mathfrak{g}_a)\mathrm{ad}(\mathfrak{g}_b)$ maps $\mathfrak{g}_c$ onto $\mathfrak{g}_{c+a+b}$ by proposition 2.3.2, so when this transformation is written as a matrix whose basis is compatible with the root-space decomposition each diagonal entry will be zero. Hence, $B(\mathfrak{g}_a, \mathfrak{g}_b) = \mathrm{Tr}(\mathrm{ad}(\mathfrak{g}_a)\mathrm{ad}(\mathfrak{g}_b)) = 0$.

(ii) By Cartan's criterion for semisimplicity, (1.4.2), B is non-degenerate on $\mathfrak{g}$, in particular the sets $B(X, \mathfrak{g})$ are all nontrivial for $X \in \mathfrak{g}_a \setminus \{0\}$. Part (i) shows that $B(X, \mathfrak{g}_b) = 0$ for all $b \neq -a$ so $B(X, \mathfrak{g}_{-a}) \neq 0$.

(iii) Suppose $-a \notin \Phi$ then $\mathfrak{g}_{-a} = 0$ and hence $B(\mathfrak{g}_a, \mathfrak{g}_{-a}) = 0$ which contradicts part (ii).

(iv) By (ii), $B(X, \mathfrak{g}_0) \neq 0$ for all $X \in \mathfrak{g}_0 \setminus \{0\}$, but $\mathfrak{g}_0 = \mathfrak{h}$ so $B|_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate.

(v) If $H \in \mathfrak{h}$ is such that $a(H) = 0$ for all $a \in \Phi$, then the root-space decomposition tells us that $\mathrm{ad}(H)$ is nilpotent on $\mathfrak{g}$ by the definition of the $\mathfrak{g}_a$ spaces. Since $\mathfrak{h}$ is abelian, $\mathrm{ad}(H)\mathrm{ad}(H')$ is nilpotent for all $H' \in \mathfrak{h}$.
By Engel's theorem (1.3.9), $B(H, \mathfrak{h}) = \mathrm{Tr}(\mathrm{ad}(H)\mathrm{ad}(\mathfrak{h})) = 0$ so by part (iv), $H = 0$ and therefore Φ spans $\mathfrak{h}^*$. □

By part (iv) of this proposition we can see that for each $a \in \Phi$ there is some $H_a \in \mathfrak{h}$ such that $a(H) = B(H, H_a)$ for all $H \in \mathfrak{h}$. Also, by proposition 2.3.2 we may choose a root vector E_a in $\mathfrak{g}_a$ with the property that $[H, E_a] = a(H)E_a$ for all $H \in \mathfrak{h}$. Later in the chapter we will show that the subspace of $\mathfrak{g}$ with basis $\{E_a, E_{-a}, H_a\}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$.

Lemma 2.3.6 *Let $\mathfrak{g}$ be a complex semisimple Lie group and let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$. Let E_a, and H_a be as defined above.*

(i) *If $a \in \Phi$ and $X \in \mathfrak{g}_{-a}$ then $[E_a, X] = B(E_a, X)H_a$.*

(ii) *If $a, b \in \Phi$ then $b(H_a)$ is a rational multiple of $a(H_a)$.*

(iii) *If $a \in \Phi$ then $a(H_a) \neq 0$.*

Proof:

(i) By proposition 2.3.5(iii), $[\mathfrak{g}_a, \mathfrak{g}_{-a}] \subseteq \mathfrak{g}_0$, so $[E_a, X] \in \mathfrak{h}$.

For $H \in \mathfrak{h}$,

$$\begin{aligned} B(([E_a, X], H) &= B([X, [H, E_a]) = a(H)B(X, E_a) \\ &= B(H, H_a)B(X, E_a) = B(H_a, H)B(E_a, X) \\ &= B(B(E_a, X)H_a, H). \end{aligned}$$

Therefore, $B([E_a, X] - B(E_a, X)H_a, H) = 0$ for all $H \in \mathfrak{h}$.

By proposition 2.3.5(iv) it follows that $[E_a, X] - B(E_a, X, H_a) = 0$ as required.

(ii) By proposition 2.3.5(ii) there is some $X_{-a} \in \mathfrak{g}_{-a}$ such that $B(E_a, X_{-a}) = 1$, so by part (i), $[E_a, X_{-a}] = H_a$.

Fix $b \in \Phi$ and let $\mathfrak{g}' = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_{b+na}$. This subspace is invariant under $\mathrm{ad}(H_a)$. We will now compute the trace of $\mathrm{ad}(H_a)$ in two ways.

Note $\mathrm{ad}(H_a)$ acts on $\mathfrak{g}_{b+na}$ with the single generalised eigenvalue $(b + na)(H_a)$ and summing the trace of all values of n we obtain the result

$$\mathrm{Tr}(\mathrm{ad}(H_a)) = \sum_{n \in \mathbb{Z}} (b + na)(H_a) \dim(\mathfrak{g}_{b+na})(\dagger)$$

as the trace is equal to the sum of the eigenvalues and the multiplicity of the eigenvalue $(b+na)(H_a)$ is $(\dim \mathfrak{g}_{b+na})$

On the other hand, by proposition 2.3.5(iii), $\mathfrak{g}'$ is invariant under $\mathrm{ad}(E_a)$ and $\mathrm{ad}(X_{-a})$. Using the fact $[E_a, X_{-a}] = H_a$ we see that

$$\mathrm{Tr}(\mathrm{ad}(H_a)) = \mathrm{Tr}(\mathrm{ad}(E_a)\mathrm{ad}(X_{-a}) - \mathrm{ad}(X_{-a})\mathrm{ad}(E_a)) = 0.$$

Thus $\dagger = 0$, so as there are only finitely many eigenvalues the result holds.

(iii) If $a(H_a) = 0$ then by part (ii) $b(H_a) = 0$ for all $b \in \Phi$. Proposition 2.3.5(v) says that every member of $\mathfrak{h}^*$ vanishes on H_a and therefore $H_a = 0$, which contradicts 2.3.5(iv) as $a \neq 0$. □

The following theorem shows that the example $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$ is more similar to the general case than may be expected.

Proposition 2.3.7 *Let $a \in \Phi$, then $\dim(\mathfrak{g}_a) = 1$. Also $na \notin \Phi$ for any integer $n \geq 2$.*

Proof: Again choose $X_{-a} \in \mathfrak{g}_{-a}$ with $B(E_a, X_{-a}) = 1$ and notice that $[E_a, X_{-a}] = H_a$.

Set

$$\mathfrak{g}'' = \mathbb{C}E_a \oplus \mathbb{C}H_a \oplus \bigoplus_{n<0} \mathfrak{g}_{na}.$$

Then $\mathfrak{g}''$ is invariant under $\mathrm{ad}(E_a)$ and $\mathrm{ad}(H_a)$ by proposition 2.3.5(iii) and is invariant under $\mathrm{ad}(X_{-a})$ by proposition 2.3.5(iii) and lemma 2.3.6(i). Therefore

$$\mathrm{Tr}(\mathrm{ad}(H_a)) = a(H_a) + 0 + \sum_{n<0} n.a(H_a)\dim(\mathfrak{g}_{na}) = 0,$$

so by rearranging we see that $\sum_{n=1}^{\infty} n \dim(\mathfrak{g}_{-na}) = 1$.

Consequently, $\dim(\mathfrak{g}_{-a}) = 1$ and $\dim(\mathfrak{g}_{-na}) = 0$ for all $n \geq 2$, since $a \in \Phi$ if and only if $-a \in \Phi$. □

We note the following corollary of theorem 2.3.7.

Corollary 2.3.8 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$.*

(i) *The action of $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ on $\mathfrak{g}$ is simultaneously diagonalisable.*

(ii) *$B|_{\mathfrak{h}\times\mathfrak{h}}$ is given by $B(H, H') = \sum_{a\in\Phi} a(H)a(H')$.*

(iii) *The vector pairs $\{E_a, E_{-a}\}$ may be chosen so that $B(E_a, E_{-a}) = 1$.*

Proof:

(i) $\mathfrak{h}$ is abelian and each root-space is one dimensional, so $\mathfrak{g}$ must have a simultaneous basis of eigenvectors and hence it is simultaneously diagonalisable.

(ii) Let $\{H_i\}$ be a basis for $\mathfrak{h}$, so by proposition 2.3.7, $\{H_i\} \cup \{E_a\}$ is a basis for $\mathfrak{g}$ which $\mathrm{ad}(H)$ acts on diagonally by (i). Therefore $\mathrm{ad}(H)\mathrm{ad}(H')$ acts diagonally and its eigenvalues are 0 and $a(H)a(H')$. Hence
$$B(H, H') = \mathrm{Tr}(\mathrm{ad}(H)\mathrm{ad}(H')) = \sum_{a\in\Phi} a(H)a(H').$$

(iii) By proposition 2.3.5(ii) B is non-singular on $\mathfrak{g}_a \times \mathfrak{g}_{-a}$. These spaces are one dimensional so the result holds. □

The preceding results from this section are sufficient to show that any complex semisimple Lie algebra is built out of copies of $\mathfrak{sl}_2(\mathbb{C})$ in a certain way.

To see this, let E_a, E_{-a} be the normalised root vectors given by corollary 2.3.8(iii).

Lemma 2.3.6(i) gives us the following bracket relations.

$$[H_a, E_a] = a(H_a)E_a, \qquad [H_a, E_{-a}] = -a(H_a)E_{-a} \quad \text{and} \quad [E_a, E_{-a}] = H_a$$

Then we may normalise these vectors by setting

$$H'_a = \frac{2}{a(H_a)}H_a, \qquad E'_a = \frac{2}{a(H_a)}E_a, \quad \text{and} \quad E'_{-a} = E_{-a},$$

so

$$[H'_a, E'_a] = 2E'_a, \qquad [H'_a, E'_{-a}] = -2E'_{-a}, \quad \text{and} \quad [E'_a, E'_{-a}] = H'_a$$

Define $e, f, h \in \mathfrak{sl}_2(\mathbb{C})$ as follows,

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These satisfy the bracket relations

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h$$

Consequently, using results from section 1.6 we know that a function which maps

$$H'_a \mapsto h, \quad E'_a \mapsto e, \quad \text{and} \quad E'_{-a} \mapsto f$$

can be extended linearly to an isomorphism of $\text{span}\{H_a, E_a, E_{-a}\}$ onto $\mathfrak{sl}_2(\mathbb{C})$. For this reason we will define the Lie subalgebra $\mathfrak{sl}_a := \text{span}\{H_a, E_a, E_{-a}\} \cong \mathfrak{sl}_2(\mathbb{C})$.

Thus, by the root-space decomposition, $\mathfrak{g}$ is spanned by embedded copies of $\mathfrak{sl}_2(\mathbb{C})$.

Definition 2.3.9 Root Strings

Let $a \in \Phi$ and let $b \in \Phi \cup \{0\}$. The a string containing b is the subset of $\Phi \cup \{0\}$ consisting of all elements of the form $b + na$ for some $n \in \mathbb{Z}$.

We define an bilinear form on $\mathfrak{h}^*$ by $\langle a, b\rangle = B(H_a, H_b) = a(H_b) = b(H_a)$ for $a, b \in \mathfrak{h}^*$ with H_a and H_b as defined in 2.3.5(iv). Then we can regard $(\mathfrak{h}^*, \langle \cdot, \cdot \rangle)$ as a bilinear inner product space. To investigate this further we study the action of such a copy of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathfrak{g}$. In particular we consider the adjoint representation of the copy of $\mathfrak{sl}_2(\mathbb{C})$ generated by the root $a \in \Phi$ on the invariant subspace $\mathfrak{g}' = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{b+na}$ given by the root string of a containing b.

Proposition 2.3.10 *Let $a \in \Phi, b \in \Phi \cup \{0\}$.*

(i) *The a string containing b has the form $\{b + na \mid -p \le n \le q\}$ where $p, q \ge 0$ and $p - q = \frac{2\langle b,a\rangle}{\langle a,a\rangle} \in \mathbb{Z}$,*

(ii) *If $b \neq na$ for all $n \in \mathbb{Z}$, then the adjoint representation of $\mathfrak{sl}_a$ on $\mathfrak{g}' := \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{b+na}$ is irreducible.*

Proof: Omitted, see [Kn02] pages 144-146. □

There are three key corollaries to this result, the last of which will be crucial for the proof of theorem 2.3.4.

Corollary 2.3.11 *If $a, b \in \Phi \cup \{0\}$ with $a + b \neq 0$ then $[\mathfrak{g}_a, \mathfrak{g}_b] = \mathfrak{g}_{a+b}$.*

Proof: Without loss we may assume that $a \neq 0$. By proposition 2.3.2, $[\mathfrak{g}_a, \mathfrak{g}_b] \subseteq \mathfrak{g}_{a+b}$. We consider two cases.

Case 1: $b = na$ for some $n \in \mathbb{Z} \setminus \{-1\}$, then proposition 2.3.7 shows $b = 0$ or $b = a$.

If $b = a$ then $\mathfrak{g}_{a+b} = 0$ by proposition 2.3.7 and equality must hold.
If $b = 0$ then the equality $[\mathfrak{h}, \mathfrak{g}_a] = \mathfrak{g}_a$ holds as $\mathfrak{g}_a$ does not commute with $\mathfrak{h}$ and thus the space $[\mathfrak{h}, \mathfrak{g}_a]$ is at least 1 dimensional, so must be equal to $\mathfrak{g}_a$.

Case 2: $b \neq na$ for all $n \in \mathbb{Z}$, then by proposition 2.3.10(ii), $\mathfrak{sl}_a$ acts irreducibly on $\mathfrak{g}' = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{b+na}$. The result then follows from theorem 1.6.2. □

Corollary 2.3.12 *Let $a, b \in \Phi$ be such that $b + na \neq 0$ for all n. Let E_a, E_{-a}, E_b be any root vectors for $a, -a, b$ respectively and let p, q be the integers defined in 2.3.10(i). Then,*

$$[E_{-a}, [E_a, E_b]] = \frac{q(1-p)}{2} a(H_a) B(E_a, E_{-a}) E_b.$$

Proof: Both sides of this equation are linear in E_a and E_{-a} so we may normalise these two vectors so that $B(E_a, E_{-a}) = 1$. Identifying the span of $\{H_a, E_a, E_{-a}\}$ with $\mathfrak{sl}_2(\mathbb{C})$ we can rewrite the desired formula as

$$\frac{\langle a, a\rangle}{2} [f, [e, E_b]] \stackrel{?}{=} \frac{q(1+p)}{2} a(H_a) E_b.$$

As $a(H_a) = \langle a, a\rangle$ this formula simplifies to

$$[f, [e, E_b]] \stackrel{?}{=} q(1+p) E_b.$$

By proposition 2.3.10, the adjoint representation of $\text{span}\{e, f, h\}$ on $\mathfrak{g}'$ is irreducible. Comparing these facts with theorem 1.6.2, we see that the vector E_{b+qa} corresponds to a multiple of v_0. Moreover, as E_b is a multiple of $(\text{ad}(f))^q E_{b+qa}$, E_b corresponds to a multiple of v_q. Using parts (iii) and (iv) from theorem 1.6.2, we see that

$$(\text{ad}(f))(\text{ad}(e)) E_b = q(N - q + 1) E_b, \text{ where } N = \dim(\mathfrak{g}') - 1 = (q + p + 1) - 1.$$

Therefore $q(N - q + 1) = q(1 + p)$, as required. □

Corollary 2.3.13 *Let V be the $\mathbb{R}$ linear span of Φ in $\mathfrak{h}^*$. Then V is a real form of the vector space $\mathfrak{h}^*$ and the restriction of $\langle\cdot,\cdot\rangle$ to $V \times V$ is a positive definite inner product.*
Moreover, if $\mathfrak{h}_0$ denotes the $\mathbb{R}$ linear span of all H_a for $a \in \Phi$ then $\mathfrak{h}_0$ is a real form of $\mathfrak{h}$ and the members of V are exactly those linear functionals that are real on $\mathfrak{h}_0$ and the restriction of those linear functionals from $\mathfrak{h}$ to $\mathfrak{h}_0$ is an isomorphism of real vector spaces from V onto $(\mathfrak{h}_0)^*$.

Proof: Omitted, see [Kn02] pages 147-148.

Definition 2.3.14 *Let $a \in \Phi$, the root reflection $s_a : (\mathfrak{h}_0)^* \to (\mathfrak{h}_0)^*$ is given by*

$$s_a(b) = b - \frac{2\langle b,a\rangle}{\langle a,a\rangle}a \quad \text{for any } b \in (\mathfrak{h}_0)^*.$$

The root reflection is an orthogonal transformation on $(\mathfrak{h}_0)^*$, $s_a(ka) = -ka$ for all $k \in \mathbb{R}$ and $s_a(b) = b$ whenever b is in the orthogonal component of a.

The subgroup $W = W(\mathfrak{g},\mathfrak{h})$ of the orthogonal group on $(\mathfrak{h}_0)^*$ generated by the set of root reflections $S := \{s_a \mid a \in \Phi\}$ is called the Weyl group of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$.

Proposition 2.3.15 *Let $a, b \in \Phi$, then $s_a(b) \in \Phi$.*

Proof: Let p, q be as defined in proposition 2.3.10. Then

$$s_a(b) = b - \frac{2\langle b,a\rangle}{\langle a,a\rangle}a = b - (p-q)a = b + (q-p)a$$

Since $-p \leq q - p \leq q$, $b + (q-p)a$ is in the a string containing b. Hence $s_a(b) \in \Phi \cup \{0\}$. Because s_a is an orthogonal transformation, $s_a(b) \neq 0$, so $s_a(b) \in \Phi$. □

Proof of theorem 2.3.4: By corollary 2.3.13, $(\mathfrak{h}_0)^*$ is a bilinear inner product space spanned by Φ. Part (ii) of the definition follows from proposition 2.3.15 and part (iii) by proposition 2.3.10(i). Finally by proposition 2.3.7, Φ is reduced. □

2.4 Abstract Root Systems

We now know that every pair $(\mathfrak{g},\mathfrak{h})$ admits a reduced abstract root system $((\mathfrak{h}_0)^*, \Phi)$. In this section we will show how each abstract root system defines an abstract Cartan matrix and how this matrix can be used to form a Dynkin diagram.

Definition 2.4.1 **Strings**

Let Φ be a reduced abstract root system. If $a \in \Phi$ and $b \in \Phi \cup \{0\}$, then the a string containing b is the set $\{\phi \in \Phi \mid \phi = b + na \text{ for some } n \in \mathbb{Z}\}$.

Proposition 2.4.2 *Let Φ be a reduced abstract root system.*

(i) *If $a \in \Phi$ then $-a \in \Phi$.*

(ii) *If $a \in \Phi$ then the only members of Φ proportional to a are $\pm a$.*

(iii) *If $a, b \in \Phi$, with $b \neq \pm a$ then $\frac{2\langle b,a\rangle}{\langle a,a\rangle} \in \{0, \pm 1, \pm 2, \pm 3\}$.*

(iv) *If $a, b \in \Phi$ are nonproportional with $\langle a,a\rangle < \langle b,b\rangle$ then $\frac{2\langle b,a\rangle}{\langle b,b\rangle} \in \{0, \pm 1\}$.*

(v) *If $a, b \in \Phi$ and $\langle a,b\rangle > 0$ then $a - b \in \Phi \cup \{0\}$, moreover, if $\langle a,b\rangle < 0$ then $a + b \in \Phi \cup \{0\}$.*

(vi) *If $a, b \in \Phi$ and $a - b, a + b \notin \Phi \cup \{0\}$, then $\langle a,b\rangle = 0$.*

(vii) *If $a \in \Phi$ and $b \in \Phi \cup \{0\}$ then the a string containing b has at most 4 elements and equals the set*

$$\{b + na \mid -p \leq n \leq q\} \text{ where } p - q = \frac{2\langle b,a\rangle}{\langle a,a\rangle}.$$

Proof:

(i) $s_a(a) = a - \frac{2\langle a,a\rangle}{\langle a,a\rangle} a = a - 2a = -a$, so $-a \in \Phi$.

(ii) Let $a \in \Phi$ and let $ca \in \Phi$. Then

$$\frac{2\langle ca, a\rangle}{\langle a, a\rangle} = 2c \in \mathbb{Z} \text{ and } \frac{2\langle a, ca\rangle}{\langle ca, ca\rangle} = \frac{2}{c} \in \mathbb{Z}.$$

As Φ is reduced $c \neq 2^{\pm 1}$, hence $c = \pm 1$.

(iii) By the Schwarz inequality on inner products

$$\left|\frac{2\langle b, a\rangle}{\langle a, a\rangle}\right| \left|\frac{2\langle a, b\rangle}{\langle b, b\rangle}\right| \leq 4$$

with equality if and only if $b = ca$ for some c. This case is covered by part (ii).

If the inequality is strict then

$$\left|\frac{2\langle b, a\rangle}{\langle a, a\rangle}\right| \left|\frac{2\langle a, b\rangle}{\langle b, b\rangle}\right| \leq 3.$$

(iv) In this case $\left|\frac{2\langle b,a\rangle}{\langle a,a\rangle}\right| \geq \left|\frac{2\langle a,b\rangle}{\langle b,b\rangle}\right|$ and their product is at most three, so $\frac{2\langle a,b\rangle}{\langle b,b\rangle} \in \{0, \pm 1\}$.

(v) The cases $b = \pm a$ are both trivial, so assume a and b are not proportional.

If $\langle a, a\rangle \leq \langle b, a\rangle$ then $s_b(a) = a - \frac{2\langle a,b\rangle}{\langle b,b\rangle} b = a - b$ by part (iv)

If instead $\langle a, a\rangle \geq \langle b, a\rangle$ then $s_a(b) = b - a$ so $a - b \in \Phi$ by part (i). To prove the second part simply replace a by $-a$ in this argument.

(vi) This follows immediately from part (v).

(vii) Let $-p$ and q be the smallest and largest values of n such that $b + na \in \Phi \cup \{0\}$. If this set has a gap then there exist values r and s such that $r < s - 1$ and $b + ra,\ b + sa \in \Phi \cup \{0\}$, with $b + (r + 1)a,\ b + (s - 1)a \notin \Phi \cup \{0\}$.

By part (v), $\langle b + ra, a\rangle \geq 0$ and $\langle b + sa, a\rangle \leq 0$.
Subtracting these inequalities we see that $(r-s)\langle a, a\rangle \geq 0$ and thus $r-s \geq 0$ which is a contradiction.

Considering the root reflection $s_a(b + na) = b + na - \frac{2\langle b+na,a\rangle}{\langle a,a\rangle} a = b - \left(n + \frac{2\langle b,a\rangle}{\langle a,a\rangle}\right) a$
we deduce that the inequality $-p \leq n \leq q$ implies that $-q \leq n + \frac{2\langle b,a\rangle}{\langle a,a\rangle} \leq p$.

Setting $n = -p$ and $n = q$ in turn we see that $\frac{2\langle b,a\rangle}{\langle a,a\rangle} \leq p - q \leq \frac{2\langle b,a\rangle}{\langle a,a\rangle}$.

Thus they are equal. It remains to calculate the length of the string. We may assume $q = 0$ so the length of the string is $p + 1$, where $p = \frac{2\langle b,a\rangle}{\langle a,a\rangle} \leq 3$ by part (iii). □

Definition 2.4.3 Positive elements

We define a notion of positivity on $(\mathfrak{h}_0)^*$ *by fixing a basis* $\{H_1, \ldots, H_m\}$ *for* $\mathfrak{h}_0$*. Then we say* $\phi > 0$ *if there is some* k *such that* $\phi(H_i) = 0$ *for all* $i < k$ *and* $\phi(H_k) > 0$*.*

We can then extend $>$ *to an ordering on* V *where* $a > b$ *if and only if* $a - b > 0$*.*

Such an ordering is often called lexicographic.

Lemma 2.4.4 *If* a, b *are distinct simple roots then* $a - b$ *is not a root, i.e.* $\langle a, b\rangle \leq 0$*.*

Proof: Assume $a - b$ is a root. If $a - b > 0$ then $a = (a - b) + b$ so a is not a simple root. If instead $a - b < 0$ then $b = a + (b - a)$ so b is not simple. Finally, if $a - b = 0$ then a, b are not distinct. Thus $a - b$ is not a root so by proposition 2.4.2(v), $\langle a, b\rangle \leq 0$. □

Proposition 2.4.5 A basis of simple roots

Set $m = \dim(\mathfrak{h}_0)^*$. There are m simple roots $a_1, \dots, a_m$ and they are linearly independent.

Moreover, if $b \in \Phi$ and $b = \sum_{i=1}^{l} c_i a_i$ then each $c_i \in \mathbb{Z}$ and either $c_i \geq 0$ for all i or $c_i \leq 0$ for all i.

Proof: Let $b > 0$ be in Φ. If b is not simple then write $b = b_1 + b_2$ with $b_1,\ b_2 > 0$. Continue decomposing until b is written as the sum of simple roots. We can list the decomposition as tuples $(b,\ b_1,$ component of $b_1, \dots)$ with each entry a component of the preceding one.

No tuple has more entries than there are positive roots, (if a root appeared twice we would get some simple root a twice, implying that $a = a + \phi$ for some sum of positive roots ϕ). Thus $b = \sum_{i=1}^{m} c_i a_i$ where each c_i is a non-negative integer and all the a_i are simple roots. Hence the set of simple roots spans Φ.

To prove linear independence suppose $c_1 a_1 + \cdots + c_s a_s - c_{s+1} a_{s+1} - \cdots - c_m a_m = 0$ with all $c_j \in \mathbb{R}_0^+ = \{x \in \mathbb{R} \,|\, x \geq 0\}$.

Set $b = c_1 a_1 + \cdots + c_s a_s = c_{s+1} a_{s+1} + \cdots + c_m a_m$. Then

$$0 \leq \langle b, b \rangle = \left\langle \sum_{j=1}^{s} c_j a_j, \sum_{k=s+1}^{m} c_k a_k \right\rangle = \sum_{j=1}^{s} \sum k = 1^m c_j c_k \langle a_j, a_k \rangle \leq 0,$$

where the last inequality holds by lemma 2.4.4. Hence $\langle b, b \rangle = 0$ so $b = 0$ and $c_j = 0$ for all j since a positive combination of positive roots cannot equal zero. □

As Φ spans $(\mathfrak{h}_0)^*$ it is clear that the set $\{a_1, \dots, a_m\}$ is a basis of $(\mathfrak{h}_0)^*$.

For the remainder of the section we fix a reduced root system Φ and an ordering $<$ on Φ. Let Π be the set of simple roots with respect to $<$.

Definition 2.4.6 Cartan matrices

Write $\Pi = \{a_1, \dots a_n\}$. The $n \times n$ matrix $A = (A_{ij})$ given by $A_{ij} = \frac{2\langle a_j, a_i \rangle}{\langle a_i, a_i \rangle}$ is called the Cartan matrix of Φ with respect to Π.

Proposition 2.4.7 Properties of the Cartan matrix

Let A be the Cartan matrix of Φ with respect to Π. Then

(i) *$A_{ij} \in \mathbb{Z}$ for all i, j,*

(ii) *$A_{ii} = 2$ for all i,*

(iii) *$A_{ij} \leq 0$ for $i \neq j$,*

(iv) *$A_{ij} = 0$ if and only if $A_{ji} = 0$,*

(v) *There is a diagonal matrix D with positive entries such that DAD^{-1} is symmetric positive definite.*

Remark: Any matrix which satisfies the five properties above is called an abstract Cartan matrix and two abstract Cartan matrices A, B are said to be isomorphic if there is a permutation matrix P such that $PAP^{-1} = B$.

Proof: (i)-(iv) are all obvious from the definition.

(v) Set $D = (D_{ij})$ where $D_{ij} = 0$ whenever $i \neq j$ and $D_{ii} = |a_i| := \langle a_i, a_i \rangle^{\frac{1}{2}}$, then

$$DAD^{-1} = \left(2 \left\langle \frac{a_i}{|a_i|}, \frac{a_j}{|a_j|} \right\rangle \right)_{ij.}$$

This matrix is positive definite as given any $c = (c_1, \dots, c_n)$,

$$c(\langle \phi_i, \phi_j \rangle) c^T = \left\langle \sum c_i \phi_i, \sum c_i \phi_i \right\rangle > 0, \quad \text{for any basis } \{\phi_1, \dots, \phi_n\} \text{ of } (\mathfrak{h}_0)^*.$$

By proposition 2.4.5, the vectors $\left\{ \frac{a_1}{|a_1|}, \dots, \frac{a_n}{|a_n|} \right\}$ form a basis of $(\mathfrak{h}_0)^*$ so the result holds. □

Proposition 2.4.8 *Let A be an abstract Cartan matrix.*

(i) *The matrix A^k obtained from A by deleting the k^{th} row and column is also an abstract Cartan matrix.*

(ii) *If $i \neq j$, then $A_{ij}A_{ji} < 4$ and $A_{ij} \in \{0, -1, -2, -3\}$.*

Proof: Part (i) follows from the definition by deleting the same row and column from the matrix D.

Let A' be the abstract Cartan matrix obtained from A by deleting all but the i^{th} and j^{th} rows and columns. As A' is an abstract Cartan matrix there is a diagonal matrix D' such that $D'A'(D')^{-1}$ is positive definite, so in particular

$$\det \begin{pmatrix} d_i & 0 \\ 0 & d_j \end{pmatrix} \begin{pmatrix} 2 & A_{ij} \\ A_{ji} & 2 \end{pmatrix} \begin{pmatrix} d_i^{-1} & 0 \\ 0 & d_j^{-1} \end{pmatrix} > 0$$

Therefore $A_{ij}A_{ji} < 4$. $A_{ij} = 0$ if and only if $A_{ji} = 0$ and $A_{ij}, A_{ji} \leq 0$ are integers, so $A_{ij} \in \{0, -1, -2, -3\}$. □

Definition 2.4.9 Dynkin diagrams

Let $A = (A_{ij})$ be an $n \times n$ abstract Cartan matrix. The Dynkin diagram of A is the graph on n vertices $\{A_1, \dots, A_n\}$ defined as follows.

Each vertex A_i has an attached weight $c\langle A_{ii}, A_{ii}\rangle$ where $c > 0$ is a constant independent of i. c is chosen minimally such that every weight takes an integer value. Given two distinct vertices A_i and A_j we connect them with an edge if $A_{ij}A_{ji} \neq 0$ and attach a weight w_{ij} to that edge, where

$$w_{ij} = \begin{cases} 3, & \text{if } A_{ij}A_{ji} = 1 \\ 4, & \text{if } A_{ij}A_{ji} = 2 \\ 6, & \text{if } A_{ij}A_{ji} = 3. \end{cases}$$

Conventionally, edge labels of value 3 are omitted from the Dynkin diagram. Also, vertex weights are often omitted, replaced by directed edges when adjacent roots have distinct weights, always directed towards the shorter root. Figure 2.1.2 includes both of these pieces of information for clarity. Another common way of labelling the edges is to draw $A_{ij}A_{ji}$ edges between the two vertices A_i and A_j.

Notice that given any abstract Dynkin diagram, we may recover the abstract Cartan matrix A which defined it. Given any two vertices A_i, A_j with no edge between them, we deduce that $A_{ij}A_{ji} = 0$ and since zeroes are symmetric in the Cartan matrix, $A_{ij} = A_{ji} = 0$.

Now suppose there is an edge between A_i and A_j where the weight of the vertices A_i and A_j are w_i and w_j respectively. The weight of the edge on the Dynkin diagram tells us the value of $A_{ij}A_{ji}$ and since DAD^{-1} is symmetric, we see that

$$(w_i)^{\frac{1}{2}} A_{ij} (w_j)^{-\frac{1}{2}} = (w_j)^{\frac{1}{2}} A_{ji} (w_i)^{\frac{1}{2}}.$$

In other words, $w_i A_{ij} = w_j A_{ji}$. These two pieces of information, added to the fact that $A_{ij}, A_{ji} \in \{-1, -2, \dots\}$, are sufficient to deduce the values A_{ij} and A_{ji}.

Theorem 2.4.10 Complex simple Lie algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. $\mathfrak{g}$ is simple if and only if its Dynkin diagram is connected.

The condition that the Dynkin diagram is connected is equivalent to requiring that any the Cartan matrix cannot be decomposed into a block diagonal matrix with more than one block and this is equivalent to requiring that the root system $\Phi(\mathfrak{g}, \mathfrak{h})$ is irreducible, so it does not admit a nontrivial orthogonal decomposition. We will implicitly use the characterisation of semisimple Lie algebras (theorem 1.4.4) in the proof.

Proof: Suppose $\mathfrak{g}$ is a nontrivial direct sum of ideals $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$. Let $a \in \Phi$ and decompose the corresponding root vector $E_a = E_a' + E_a''$. Let $H \in \mathfrak{h}$, then

$$0 = [H, E_a] - a(H)E_a = ([H, E_a'] - a(H)E_a') + ([H, E_a''] - a(H)E_a'').$$

$\mathfrak{g}'$ and $\mathfrak{g}''$ are both ideals of $\mathfrak{g}$ and have trivial intersection, so the two terms on the right hand side of this equation must both by 0. Thus E_a' and E_a'' are both contained in the root-space $\mathfrak{g}_a$. As $\dim(\mathfrak{g}_a) = 1$, by proposition 2.3.7, $E_a' = 0$ or $E_a'' = 0$. Thus $\mathfrak{g}_a \subseteq \mathfrak{g}'$ or $\mathfrak{g}_a \subseteq \mathfrak{g}''$.

Define

$$\Phi' = \{a \in \Phi \,|\, \mathfrak{g}_a \subseteq \mathfrak{g}'\} \quad \text{and} \quad \Phi'' = \{a \in \Phi \,|\, \mathfrak{g}_a \subseteq \mathfrak{g}''\}.$$

Therefore Φ is the disjoint union of Φ' and Φ''. Finally,

$$a'(H_{a''})E_{a'} = [H_{a''}, E_{a'}] \in [H_{a''}, \mathfrak{g}'] = [[E_{a''}, E_{-a''}], \mathfrak{g}'] \subseteq [\mathfrak{g}'', \mathfrak{g}'] = 0.$$

Thus $a'(H_{a''}) = 0$, so Φ' and Φ'' are mutually orthogonal.

Suppose Φ is not irreducible, so $\Phi = \Phi' \cup \Phi''$ exhibits Φ as a nontrivial union of two mutually orthogonal (and hence disjoint) sets. Define

$$\mathfrak{g}' = \sum_{a \in \Phi'} \{\mathbb{C}H_a + \mathfrak{g}_a + \mathfrak{g}_{-a}\} \text{ and } \mathfrak{g}'' = \sum_{a \in \Phi''} \{\mathbb{C}H_a + \mathfrak{g}_a + \mathfrak{g}_{-a}\}.$$

Then $\mathfrak{g}'$ and $\mathfrak{g}''$ are both subspaces of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ as a vector space. To complete the proof it suffices to show that $\mathfrak{g}'$ and $\mathfrak{g}''$ are both ideals of $\mathfrak{g}$. It is apparent that both are closed under the Lie bracket, so are Lie subalgebras of $\mathfrak{g}$.

As Φ' and Φ'' are mutually orthogonal $[H_{a'}, E_{a''}] = a''(H_{a'})E_{a''} = 0$ for all $a' \in \Phi'$ and $a'' \in \Phi''$.

Suppose $[\mathfrak{g}_{a'}, \mathfrak{g}_{a''}] \neq 0$. Then the root $a' + a''$ is not orthogonal to a' or to a'', contradicting the decomposition of Φ. Thus $[\mathfrak{g}_{a'}, \mathfrak{g}_{a''}] = 0$.

Combining these two results, we see that $[\mathfrak{g}', \mathfrak{g}_{a''}] = 0 = [\mathfrak{g}'', \mathfrak{g}_{a'}]$, thus $[\mathfrak{g}', \mathfrak{g}] \subseteq \mathfrak{g}'$ and $[\mathfrak{g}'', \mathfrak{g}] \subseteq \mathfrak{g}''$ as required. □

There are two remarks to make about this proof. Firstly, theorem 2.4.10 also says that $\mathfrak{g}$ is simple if and only if its Weyl group is not equal to the direct product of the Weyl groups of two other non-trivial complex semisimple Lie algebras. The second remark is that this result proves that $\mathfrak{sl}_{n+1}(\mathbb{C})$ is indeed a simple Lie algebra, as was stated in section 2.2.

2.5 Classification of connected abstract Dynkin diagrams

Theorem 2.5.1 Classification

Up to isomorphism the connected abstract Dynkin diagrams are precisely those in figure 2.1.2, namely,

(i) A_n *for* $n \geq 1$,

(ii) B_n *for* $n \geq 2$,

(iii) C_n *for* $n \geq 3$,

(iv) D_n *for* $n \geq 4$,

(v) E_6, E_7, E_8, F_4 *and* G_2

We have constructed many of the results needed to prove this, however, we will not give the full proof but will instead demonstrate how the results we have already obtained allow us to restrict the possibilities substantially.

Trivially, yet also importantly, a connected abstract Dynkin diagram has finite vertex set.

Given a connected Dynkin diagram whose ith and jth vertices are connected by a single edge, (so have the same weight), we can obtain a new Dynkin diagram by collapsing these two vertices down to a single vertex, giving it the common weight and retaining all edges with exactly one end vertex from i and j. Using the fact that $A_{ij}A_{ji} \leq 3$, this construction allows us to deduce that G_2 is the only connected Dynkin diagram with an edge labelled 6.

Using the fact that $A_{ij}A_{ji} \leq 3$, we can show that a Dynkin diagram on n vertices contains at most $n-1$ pairs of adjacent vertices (vertices with an edge connecting them).

Let $a = \sum_{i=1}^{n} |a_i|^{-1} a_i$. Then

$$\begin{aligned} 0 < |a|^2 &= \textstyle\sum_{i,j} \left\langle \frac{a_i}{|a_i|}, \frac{a_j}{|a_j|} \right\rangle \\ &= \textstyle\sum_{i} \left\langle \frac{a_i}{|a_i|}, \frac{a_i}{|a_i|} \right\rangle + 2\sum_{i<j} \left\langle \frac{a_i}{|a_i|}, \frac{a_j}{|a_j|} \right\rangle \\ &= n + \textstyle\sum_{i<j} \frac{2\langle a_i, a_j \rangle}{|a_i||a_j|} \\ &= n - \textstyle\sum_{i<j} (A_{ij}A_{ji})^{\frac{1}{2}}. \end{aligned}$$

Whenever $(A_{ij}A_{ji})^{\frac{1}{2}} \neq 0$ it is at least 1. We therefore deduce that $n - \sum_{i,j \text{ adjacent}} 1 \geq 0$, so each connected Dynkin diagram is a tree.

These observations significantly limit the possibilities for a connected Dynkin diagram. One more crucial observation is that no vertex sends out more than three edges (where the Dynkin diagram is labelled by multiple edges). A complete proof of theorem 2.5.1 can be found in [Kn02] on pages 170-179.

To illustrate the theory we give two further examples which, together with section 2.2, give explicit complex simple Lie algebras corresponding to each of the four infinite families of connected Dynkin diagrams, A_n, B_n, C_n and D_n.

2.6 Other examples: $\mathfrak{so}_n(\mathbb{C})$ and $\mathfrak{sp}_n(\mathbb{C})$

Recall that the special orthogonal and symplectic Lie algebras are given by

$$\mathfrak{so}_n(\mathbb{C}) := \{X \in M_n(\mathbb{C}) \mid X + X^t = 0\} \quad \text{and}$$

$$\mathfrak{sp}_n(\mathbb{C}) := \{X \in M_{2n}(\mathbb{C}) \mid X^t J + JX = 0\} \quad \text{where } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We will show that, for suitable n, $\mathfrak{so}_{2n+1}(\mathbb{C})$ is a complex simple Lie algebra of type B_n, $\mathfrak{sp}_n(\mathbb{C})$ is a complex simple Lie algebra of type C_n and $\mathfrak{so}_{2n}(\mathbb{C})$ is a complex simple Lie algebra of type D_n.

We begin with $\mathfrak{so}_{2n+1}(\mathbb{C})$, for $n \geq 2$. The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}_{2n+1}(\mathbb{C})$ is the subalgebra containing all matrices of the form

$$H = \begin{pmatrix} 0 & ih_1 & & & & & & 0 \\ -ih_1 & 0 & & & & & & \\ & & 0 & ih_2 & & & & \\ & & -ih_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & ih_n & \\ & & & & & -ih_n & 0 & \\ 0 & & & & & & & 0 \end{pmatrix}$$

We define the linear functionals e_j so that $e_j(H) = h_j$, for $1 \leq j \leq n$. $\mathfrak{h}_0$ is the subalgebra of $\mathfrak{h}$ in which $h_i \in \mathbb{R}i$ for all i.

The set of roots with respect to this Cartan subalgebra is $\Phi = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm e_k \mid 1 \leq k \leq n\}$.

It is clear by construction that $\{e_k \,|\, 1 \leq k \leq n\}$ is an orthogonal basis for the real vector space $(\mathfrak{h}_0)^*$ in which every element has the same length. Thus, every element $\phi \in (\mathfrak{h}_0)^*$ can be written as $\sum_{i=1}^{n} c_i e_i$.

We define an element of $(\mathfrak{h}_0)^*$ to be positive if the first non-zero c_i is positive. With respect to this ordering, the following is a suitable choice of simple roots.

$$\Pi := \{e_i - e_{i+1} \,|\, 1 \leq i \leq n-1\} \cup \{e_n\}.$$

Direct calculations show that the vertices $\{e_i - e_{i+1} \,|\, 1 \leq i \leq n-1\}$ behave exactly as they do for the A_n example, so they form a path in the Dynkin diagram in which every vertex has weight 2 and each edge has label 3. The vertex e_n has weight 1, so in this case we choose the constant $c = 1$ and it is clear that only the vertex $e_{n-1} - e_n$ can form an edge with the vertex e_n. Moreover,

$$\frac{2\langle e_{n-1} - e_n, e_n\rangle}{2\langle e_{n-1} - e_n, e_{n-1} - e_n\rangle} \cdot \frac{2\langle e_{n-1} - e_n, e_n\rangle}{2\langle e_n, e_n\rangle} = 2\left(\frac{-1}{2}\right) 2\left(\frac{-1}{1}\right) = 2.$$

So the edge connecting $e_{n-1} - e_n$ to e_n is labelled 4. Thus $\mathfrak{so}_{2n+1}(\mathbb{C})$ has Dynkin diagram B_n, as claimed.

Since the Dynkin diagram of $\mathfrak{so}_{2n+1}(\mathbb{C})$ is connected, we deduce that it is a complex simple Lie algebra, by theorem 2.4.10.

Next we consider $\mathfrak{sp}_n(\mathbb{C})$, for $n \geq 3$. It can be shown that the set, $\mathfrak{h}$, of all matrices of the form

$$H = \begin{pmatrix} h_1 & & & & & 0 \\ & \ddots & & & & \\ & & h_n & & & \\ & & & -h_1 & & \\ & & & & \ddots & \\ 0 & & & & & -h_n \end{pmatrix}$$

is a maximal abelian subalgebra of $\mathfrak{sp}_n(\mathbb{C})$, so is a Cartan subalgebra. As with the previous example we define $e_j(H) = h_j$ where H is the matrix given above. We define $\mathfrak{h}_0$ to be the set of all matrices in $\mathfrak{h}$ where each $h_i \in \mathbb{R}$. Again, $\{e_k \,|\, 1 \leq k \leq n\}$ is an orthogonal basis for the real vector space $(\mathfrak{h}_0)^*$ in which every element has the same length. We use this basis to define an ordering on $(\mathfrak{h}_0)^*$ in the same way as for $\mathfrak{so}_{2n+1}(\mathbb{C})$.

The set of roots with respect to this Cartan subalgebra is $\Phi = \{\pm e_i \pm e_j \,|\, i \neq j\} \cup \{\pm 2e_i \,|\, 1 \leq i \leq n\}$. The set $\Pi = \{e_i - e_{i+1} \,|\, 1 \leq i \leq n-1\} \cup \{2e_n\}$ is a simple root system for Φ. Direct calculations show that this defines the Dynkin diagram C_n. Again by theorem 2.4.10, we deduce that $\mathfrak{sp}_n(\mathbb{C})$ is a complex simple Lie algebra.

We finish this section with $\mathfrak{so}_{2n}(\mathbb{C})$ for $n \geq 4$. All elements of the Cartan subalgebra of $\mathfrak{so}_{2n}(\mathbb{C})$ are obtained by removing the last row and column from each matrix in the Cartan subalgebra of $\mathfrak{so}_{2n+1}(\mathbb{C})$. Unsurprisingly, we define $e_j(H) = h_j$ and $\mathfrak{h}_0$ to be the subalgebra of $\mathfrak{h}$ in which $h_i \in \mathbb{R}i$ for all i.
The set of roots with respect to this Cartan subalgebra is $\Phi = \{\pm e_i \pm e_j \,|\, i \neq j\}$. We notice that $\{e_k \,|\, 1 \leq k \leq n\}$ is an orthogonal basis for the real vector space $(\mathfrak{h}_0)^*$ in which every element has the same length and define a notion of positivity accordingly. This ordering leads to the following set of simple roots

$$\Pi = \{e_i - e_{i+1} \,|\, 1 \leq i \leq j\} \cup \{e_{n-1} + e_n\}.$$

We deduce immediately that each vertex has weight 2, so we set $c = \frac{1}{2}$. The vertices $\{e_i - e_{i+1} \,|\, 1 \leq i \leq j\} \cup \{e_{n-1} + e_n\}$ form a path of length $n-1$ in the Dynkin diagram, so it remains to consider the edges with end vertex $e_{n-1} + e_n$. Since $\langle e_{n-1} + e_n, e_{n-1} - e_n\rangle = 0$, $e_{n-1} + e_n$ is connected only to the vertex $e_{n-2} - en - 1$ and that edge is labelled 3. Thus the simple Lie algebra $\mathfrak{so}_{2n}(\mathbb{C})$ has Dynkin diagram D_n.

2.7 Existence and Uniqueness theorems

This section offers an insight into how the proof of theorem 2.1.1 is completed. So far, we have proved that every complex semisimple Lie algebra admits a Dynkin diagram and section 2.5 states that each connected component of this diagram is listed in figure 2.1.2. It remains to prove that each of the Dynkin diagrams E_6, E_7, E_8, F_4 and G_2 is in fact the Dynkin diagram of some complex semisimple Lie algebra and that two complex semisimple Lie algebras with isomorphic Dynkin diagrams are isomorphic.

The first two uniqueness theorems deal with two of the more choices that were made during the process, showing that they have no effect on the Dynkin diagram.

Theorem 2.7.1 Uniqueness theorem 1

If $\mathfrak{h}$ and $\mathfrak{h}'$ are two Cartan subalgebras of $\mathfrak{g}$ then there is some $\alpha \in \mathrm{Int}(\mathfrak{g})$ with $\alpha(\mathfrak{h}) = \mathfrak{h}'$

Proof: In section 3.3, we will prove that the real forms of $\mathfrak{h}$ and $\mathfrak{h}'$ correspond to maximal tori in the corresponding semisimple Lie group G. Maximal tori are conjugate in G, so there is some element of $\mathrm{Ad}(G)$ which performs this conjugation. $d(\mathrm{Ad}) = \mathrm{ad}$ so this map acts as an element of $\alpha \in \mathrm{Int}(\mathfrak{g})$ on $\mathfrak{g}$ and $\alpha(\mathfrak{h}) = \mathfrak{h}'$. □

Theorem 2.7.2 Uniqueness theorem 2

Different enumerations of Π lead to Cartan matrices which are conjugates by a permutation matrix. Such Cartan matrices define isomorphic Dynkin diagrams.

Proof: This follows immediately from the definitions and the construction of the Dynkin diagram. □

We now build to the next result, which essentially states that the Lie algebra is in some sense uniquely determined by the way in which the various copies of $\mathfrak{sl}_2(\mathbb{C})$ contained in the Lie algebra interact.

Definition 2.7.3 Standard generators

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h}$ be some Cartan subalgebra of $\mathfrak{g}$. Let Φ be the set of roots corresponding to this choice of $\mathfrak{h}$ and let $\Pi = \{a_1, \dots, a_l\}$ be a simple system for Φ. Let B be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ that is positive definite on the real form $\mathfrak{h}_0$ of $\mathfrak{h}$, where the roots are real. Finally let $A = (A_{ij})_{i,j=1}^{l}$ be the Cartan matrix of Φ.

The set $X = \{h_i, e_i, f_i \,|\, 1 \le i \le l\}$ is called a set of standard generators of $\mathfrak{g}$ relative to $\mathfrak{h}, \Phi, B, \Pi$ and A, where

$$h_i = \frac{2}{\langle a_i, a_i \rangle} H_{a_i} \text{ and}$$

e_i and f_i are non-zero root vectors for a_i and $-a_i$ respectively, normalised so that $B(e_i, f_i) = \dfrac{2}{\langle a_i, a_i \rangle}$.

Proposition 2.7.4 *Let X be a set of standard generators of $\mathfrak{g}$ relative to $\mathfrak{h}, \Phi, B, \Pi$ and A. X generates $\mathfrak{g}$ as a Lie algebra.*

Proof: Clearly X is contained in $\mathfrak{g}$, so its span is contained in $\mathfrak{g}$. Hence we only need to show that $\mathfrak{g} \subseteq \mathrm{span}(X)$.

The h_i's form a basis for $\mathfrak{h}$, as the a_i's form a basis of its dual space $\mathfrak{h}^*$.

Let $a = \sum_{i=1}^{l} n_i a_i$ be a positive root and let e_a be any nonzero root vector. We show $e_a \in \mathrm{span}(X)$ by using induction on the level of a (which is $\sum_{i=1}^{l} n_i$). Moreover, we show that e_a is a multiple of some iterated bracket of the e_i's. (A similar argument shows that each negative root vector f_a is a multiple of some iterated bracket of the f_i's.)

If the level is 1, then $a = a_j$ for some j, and as root-spaces are one dimensional, e_a is a multiple of e_j.

Assume the result holds for all roots of level less than n and suppose a is a root of level $n > 1$.

Since $0 < \langle a, a \rangle = \sum_{i=1}^{l} n_i \langle a, a_i \rangle$, $\langle a, a_j \rangle > 0$ for some j.

By proposition 2.4.2(v), $b = a - a_j$ is a root and by proposition 2.4.5, $b > 0$.

If e_b is a nonzero root vector for b then the induction hypothesis shows that e_b is a multiple of an iterated bracket of the e_i's. Using corollary 2.3.11, we deduce that e_a is a multiple of $[e_b, e_j]$ for some nonzero root vector e_j of a_j. □

Proposition 2.7.5 Key properties of a set of standard generators

The set $X = \{e_i, f_i, h_i \,|\, 1 \leq i \leq l\}$ obeys the following relations for all i, j.

(i) $[h_i, h_j] = 0$,

(ii) $[e_i, f_j] = \delta_{ij} h_i$,

(iii) $[h_i, e_j] = A_{ij} e_j$,

(iv) $[h_i, f_j] = -A_{ij} f_j$,

(v) $(\mathrm{ad}(e_i))^{-A_{ij}+1} e_j = 0$ *whenever* $i \neq j$ *and*

(vi) $(\mathrm{ad}(f_i))^{-A_{ij}+1} f_j = 0$ *whenever* $i \neq j$.

These relations are called the Serre relations, we will meet them again in section 7.1.

Proof:

(i) $\mathfrak{h}$ is abelian by definition.

(ii) The case $i = j$ follows by lemma 2.3.6. When $i \neq j$, $a_i - a_j$ cannot be a root by proposition 2.4.5.

(iii) $[h_i, e_j] = a_j(h_i) e_j = \frac{2}{\langle a_i, a_i \rangle} a_j(H_{a_i}) e_j = A_{ij} e_j$.

(iv) Same reasoning as (iii).

(v) When $i \neq j$ the a_i string containing a_j is given by $a_j, a_j + a_i, \ldots, a_j + q.a_i$, since $a_j - a_i$ is not a root. Hence $p = 0$ for this root string and thus $-q = p - q = \frac{2\langle a_j, a_i \rangle}{\langle a_i, a_i \rangle} = A_{ij}$.

This means that $1 - A_i j = q + 1$ and $a := a_j + (1 - A_{ij}) a_i$ is not a root. As $(\mathrm{ad}(e_i))^{-A_{ij}+1} e_j \in \mathfrak{g}_a$ the result follows.

(vi) Same reasoning as (v). □

Definition 2.7.6 Free Lie algebras

Let X be a set. A free Lie algebra on X is a pair $(\mathfrak{F}, \sigma)$ consisting of a complex lie algebra $\mathfrak{F}$ (which is infinite dimensional) and a function $\sigma : X \to \mathfrak{F}$ with the following universal property.

Whenever $\mathfrak{l}$ is a complex Lie algebra and $\psi : X \to \mathfrak{l}$ is a function, then there is a unique Lie algebra homomorphism $\overline{\psi} : \mathfrak{F} \to \mathfrak{l}$ such that $\overline{\psi} \circ \sigma = \psi$.

Proposition 2.7.7 Existence and Uniqueness of free Lie algebras

If X is a non-empty set then there is a free Lie algebra on X which is unique up to isomorphism. Moreover, the image of X in $\mathfrak{F}$ (under σ) generates $\mathfrak{F}$.

Proof: This requires the Poincaré-Birkhoff-Witt theorem, which is outside the scope of this document. For a proof see [Kn02] pages 217-229. □

Let X be a set of generators for a complex semisimple Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{h}, \Phi, B, \Pi$ and A. Let $\mathfrak{F}$ be the free Lie algebra on X and let $\mathfrak{R}$ be the ideal generated by all elements of $\mathfrak{F}$ which satisfy the Serre relations.

We obtain a unique Lie algebra homomorphism $\overline{\psi}$ of $\mathfrak{F}$ into $\mathfrak{g}$, from the natural mapping ψ between X and $\mathfrak{g}$. By proposition 2.7.5, $\mathfrak{R} \subseteq \ker \overline{\psi}$. Hence this descends to a Lie algebra homomorphism $\mathfrak{F}/\mathfrak{R} \to \mathfrak{g}$ which is onto by proposition 2.7.4. This is called the canonical homomorphism of $\mathfrak{F}/\mathfrak{R}$ onto $\mathfrak{g}$ relative to $X = \{h_i, e_i, f_i \,|\, 1 \leq i \leq l\}$.

Theorem 2.7.8 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $X = \{h_i, e_i, f_i \,|\, 1 \leq i \leq l\}$ be a set of standard generators. Let $\mathfrak{F}$ be the free Lie algebra of X and let $\mathfrak{R}$ be the ideal generated by the Serre relations. Then the canonical homomorphism from $\mathfrak{F}/\mathfrak{R}$ onto $\mathfrak{g}$ is an isomorphism.*

It is clear that by definition there is a homomorphism from $\mathfrak{F}$ to $\mathfrak{g}$ which extends the 0 map on $\mathfrak{R}$, therefore it reduces to a homomorphism from $\mathfrak{F}/\mathfrak{R}$ to $\mathfrak{g}$.

The proof requires two lemmas.

Lemma 2.7.9 *Let $A = A_{ij}$ be an abstract Cartan matrix, let $\mathfrak{F}$ be the free Lie algebra of $X = \{h_i, e_i, f_i \,|\, 1 \leq i \leq l\}$ and let $\overline{\mathfrak{R}}$ be the ideal generated by the elements of $\mathfrak{F}$ which satisfy Serre relations (i) to (iv).*

Define $\overline{\mathfrak{g}} = \mathfrak{F}/\overline{\mathfrak{R}}$ and denote the images of the generators in $\overline{\mathfrak{g}}$ by $\overline{e_i}$, $\overline{f_i}$ and *$\overline{h_i}$ respectively.*

Also, let $\overline{\mathfrak{h}} =$ span *$\{\overline{h_i} \,|\, 1 \leq i \leq l\}$ and let $\overline{\mathfrak{e}}$ and $\overline{\mathfrak{f}}$ be the subalgebras of $\overline{g}$ generated by $\{\overline{e_i} \,|\, 1 \leq i \leq l\}$ and $\{\overline{f_i} \,|\, 1 \leq i \leq l\}$ respectively.*

Then $\overline{\mathfrak{g}} = \overline{\mathfrak{h}} \oplus \overline{\mathfrak{e}} \oplus \overline{\mathfrak{f}}$.

Proof: Omitted, see [Kn02] pages 189-190.

Lemma 2.7.10 *Let $A = A_{ij}$ be an abstract Cartan matrix, let $\mathfrak{F}$ be the free Lie algebra of $X = \{h_i, e_i, f_i \,|\, 1 \leq i \leq l\}$ and let $\mathfrak{R}$ be the ideal generated by the six Serre relations.*

Define $\mathfrak{g}' = \mathfrak{F}/\mathfrak{R}$ and suppose span*$\{h_i \,|\, 1 \leq i \leq l\}$ is mapped injectively from $\mathfrak{F}$ into $\mathfrak{g}'$.*

Write h_i' for the image of h_i in $\mathfrak{g}'$. Then $\mathfrak{g}'$ is a finite dimensional complex semisimple Lie algebra and the subspace $\mathfrak{h}' =$ span *$\{h_i' \,|\, 1 \leq i \leq l\}$ is a Cartan subalgebra of $\mathfrak{g}'$.*

Finally, the linear functionals $a_j \in (\mathfrak{h}')^*$ given by $a_j(H_i) = A_{ij}$ form a simple system within the root system associated to the Cartan subalgebra $\mathfrak{h}'$ and the Cartan matrix associated to this system is A.

Proof: Omitted, see [Kn02] pages 191-196.

Proof of theorem 2.7.8: By the definition of the free Lie algebra, σ maps X to a linearly independent subset of $\mathfrak{g}$. (If not then setting $\psi = 0$ in the definition will not lead to a unique $\overline{\psi}$.)

The map $\overline{\phi} : \mathfrak{F} \to \mathfrak{g}$ can be written as the composition of maps from $\mathfrak{F}$ to $\mathfrak{g}' = \mathfrak{F}/\mathfrak{R}$ and from $\mathfrak{g}'$ to $\mathfrak{g}$, so both of these maps must be injective on span$\{h_i\}$ and span$\{h_i'\}$ respectively. Because of this we may apply lemma 2.7.10, so we know that $\mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra with a Cartan subalgebra given by span$\{h_i' \,|\, 1 \leq i \leq l\}$.

The map $\mathfrak{F} \to \mathfrak{g}$ is onto by proposition 2.7.4 and hence the map $\mathfrak{g}' \to \mathfrak{g}$ is also onto. Thus $\mathfrak{g}$ is isomorphic to a quotient of $\mathfrak{g}'$.

If $\mathfrak{a}$ is a nontrivial simple ideal in $\mathfrak{g}'$ then it follows from the definition that $\mathfrak{h}' \cap \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}'$.

$\mathfrak{h}$ is mapped injectively under the quotient map from $\mathfrak{g}'$ to $\mathfrak{g}$, so $\mathfrak{h}' \cap \mathfrak{a}$ does not map to 0 and therefore $\mathfrak{a}$ does not map to 0. Thus the homomorphism $\mathfrak{g}' \to \mathfrak{g}$ has trivial kernel and the theorem holds. □

We now proceed to the isomorphism and existence theorems which complete the proof of theorem 2.1.1.

Theorem 2.7.11 Isomorphism theorem

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be complex semisimple Lie algebras with respective Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$ and root systems Φ and Φ'.

Suppose there is a vector space isomorphism $\psi : \mathfrak{h} \to \mathfrak{h}'$ such that its transpose $\psi^t : \mathfrak{h}'^* \to \mathfrak{h}^*$ has the property $\phi^t(\Phi') = \Phi$.

For each $a \in \Phi$ set $a' = (\phi^t)^{-1}(a) \in \Phi'$ and fix a simple system Π for Φ.
For each $a \in \Pi$ select nonzero root vectors $E_a \in \mathfrak{g}_a$ and $E_{a'} \in \mathfrak{g}_{a'}$.

Then there is a unique Lie algebra isomorphism $\overline{\psi} : \mathfrak{g} \to \mathfrak{g}'$ such that $\overline{\psi}|_{\mathfrak{h}} = \psi$ and $\overline{\psi}(E_a) = E_{a'}$ for each $a \in \Pi$.

Proof of uniqueness: If $\overline{\psi_1}$ and $\overline{\psi_2}$ are two such isomorphisms then $\overline{\psi_0} = \overline{\psi_2}^{-1} \circ \overline{\psi_1}$ is an automorphism of $\mathfrak{g}$ which preserves $\mathfrak{h}$ and the root vectors for each simple root. Let $\{h_i, e_i, f_i\}$ be the triple associated to the simple root a_i, it is clear that $\overline{\psi_0}$ preserves each e_i and h_i. Moreover, $\overline{\psi_0}(f_i)$ must be a root vector for $-a_i$ and hence must be a multiple of f_i, say $c_i f_i$.

Applying $\overline{\psi_0}$ to the Serre relation $[e_i, f_i] = h_i$ tells us that $c_i = 1$. Thus $\overline{\psi_0}$ preserves a generating set for $\mathfrak{g}$ so $\overline{\psi_0}$ is the identity automorphism and $\overline{\psi_1} = \overline{\psi_2}$. □

Proof of existence: The linear map $(\psi^t)^{-1}$ is given by $(\psi^t)^{-1}(a) = a' = a \circ \psi^{-1}$. By assumption this map carries Φ to Φ' and hence maps root strings to root strings.

By proposition 2.3.10(i), $\frac{2\langle b,a\rangle}{\langle a,a\rangle} = \frac{2\langle b',a'\rangle}{\langle a',a'\rangle}$ for all $a, b \in \Phi$. (†)

Write $\Pi = \{a_1, \ldots, a_l\}$ and let $\Pi' = (\psi^t)^{-1}(\Pi) = \{a'_1, \ldots, a'_l\}$.

Define h_i and h'_i to be the respective members of $\mathfrak{h}$ and $\mathfrak{h}'$ with

$$a_j(h_i) = \frac{2\langle a_j, a_i\rangle}{\langle a_i, a_i\rangle} \text{ and } a'_j(h'_i) = \frac{2\langle a'_j, a'_i\rangle}{\langle a'_i, a'_i\rangle}.$$

By (†), $a_j(h_i) = a'_j(h'_i)$ and hence $(\psi^t)^{-1}(a_j)(h'_i) = a_j(h_i) = a_j(\psi^{-1}(h'_i))$. Therefore $\psi(h_i) = h'_i$ for all i.

Set $E_{a_i} = e_i$ and let $e'_i = E_{a'_i}$. Define $f_i \in \mathfrak{g}$ to be a root vector for $-a_i$ with $[e_i, f_i] = h_i$ and define $f'_i \in \mathfrak{g}'$ to be a root vector for $-a'_i$ with $[e'_i, f'_i] = h'_i$. Then $X = \{e_i, f_i, h_i \,|\, 1 \le i \le l\}$ and $X' = \{e'_i, f'_i, h'_i \,|\, 1 \le i \le l\}$ are standard sets of generators for $\mathfrak{g}$ and $\mathfrak{g}'$ respectively.

Let $\mathfrak{F}$ and $\mathfrak{F}'$ be the free Lie algebras of X and X' and let $\mathfrak{R}$ and $\mathfrak{R}'$ be the ideals in $\mathfrak{F}$ and $\mathfrak{F}'$ generated by the Serre relations.

Define $\varphi : X \to \mathfrak{F}'$ by $\varphi(h_i) = (h'_i)$, $\varphi(e_i) = (e'_i)$ and $\varphi(f_i) = (f'_i)$. Note that the universal mapping property of $\mathfrak{F}$ shows that φ extends to a unique Lie algebra homomorphism $\overline{\varphi} : \mathfrak{F} \to \mathfrak{F}'$. By (†), $\overline{\varphi}(\mathfrak{R}) \subseteq \mathfrak{R}'$.

Therefore $\overline{\varphi}$ descends to a Lie algebra homomorphism from $\mathfrak{F}/\mathfrak{R}$ to $\mathfrak{F}'/\mathfrak{R}'$, which we denote by $\widetilde{\varphi}$.

The canonical maps $\overline{\psi_1} : \mathfrak{F}/\mathfrak{R} \to \mathfrak{g}$ and $\overline{\psi_2} : \mathfrak{F}'/\mathfrak{R}' \to \mathfrak{g}'$ are isomorphisms by theorem 2.7.8. They also satisfy the following conditions.

(i) $\overline{\psi_1}^{-1}(h_i) = h_i \mod \mathfrak{R}$,

(ii) $\overline{\psi_1}^{-1}(E_{a_i}) = e_i \mod \mathfrak{R}$,

(iii) $\overline{\psi_2}(h'_i \mod \mathfrak{R}') = h'_i$ and

(iv) $\overline{\psi_2}(e'_i \mod \mathfrak{R}') = E'_{a_i}$.

Therefore $\widetilde{\psi} = \overline{\psi_2} \circ \widetilde{\varphi} \circ \overline{\psi_1}^{-1}$ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{g}'$ with $\widetilde{\psi}(h_i) = h'_i$ and $\widetilde{\psi}(E_{a_i}) = E'_{a_i}$ for all i. As $\psi(h_i) = h'_i$ it follows that $\widetilde{\psi}|_{\mathfrak{h}} = \psi$. By assumption $\widetilde{\psi} : \mathfrak{h} \to \mathfrak{h}'$ is an isomorphism.

Using the same argument which completed the proof of theorem 2.7.8 we can deduce that $\widetilde{\psi}$ is injective.

Finally, as $\dim(\mathfrak{g}) = \dim(\mathfrak{h}) + |\Phi| = \dim(\mathfrak{h}') + |\Phi'| = \dim(\mathfrak{g}')$ it follows that $\widetilde{\psi}$ is an isomorphism. □

Corollary 2.7.12 *If $\mathfrak{g}$ and $\mathfrak{g}'$ are two complex semisimple Lie algebras with isomorphic root systems then they are isomorphic.*

Proof: Apply theorem 2.7.11 to the map which sends the root vector E_a for each simple root a to any nonzero root vector for the corresponding simple root for $\mathfrak{g}'$. □

Theorem 2.7.13 Existence theorem

If $A = (A_{ij})$ is an abstract Cartan matrix then there is a complex semisimple Lie algebra $\mathfrak{g}$ whose root system has A as its Cartan matrix.

Proof: See [Bo05], chapter VII, theorem 4.3.1 for a proof of this result. □

Chapter 3

Semisimple Lie groups

Now that we have a classification of all complex semisimple Lie algebras we focus of their associated Lie groups. The first key result is to relate compact Lie groups to semisimple Lie groups, by constructing a compact real form for every complex semisimple Lie algebra. By constructing maximal tori, we will obtain an alternative definition of the Weyl group to the one given in the previous chapter. Together these results will allow us to give two important decompositions of semisimple Lie groups, the Cartan and Iwasawa decompositions. The second of these will be critical in the chapter on buildings. The main reference for this chapter is [Kn02] chapters 4 and 6.

3.1 Example: $SL_n(\mathbb{C})$, for $n \geq 2$

Recall that $SL_n(\mathbb{C}) = \{A \in M_n(\mathbb{C}) \mid \det(A) = 1\}$. The Lie algebra of $SL_n(\mathbb{C})$ is $\mathfrak{sl}_n(\mathbb{C})$ by example 1.2.10.

The set $\mathfrak{h}$ of all diagonal matrices in $\mathfrak{g} := \mathfrak{sl}_n(\mathbb{C})$ is a Cartan subalgebra of $\mathfrak{g}$ so applying the exponential function to such matrices, we see that the set T of all diagonal matrices in $G := SL_n(\mathbb{C})$ is a maximal abelian subgroup of G, as a strictly larger maximal abelian subgroup would generate a strictly larger abelian subalgebra, which is impossible as $\mathfrak{h}$ is maximal by inclusion. The normaliser of T in G is precisely the set of monomial matrices in G (matrices of determinant 1 which have exactly one non-zero entry on each row and each column). The quotient group N/T is generated by the cosets

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} N, \quad \begin{pmatrix} 1 & & & & \\ & 0 & 1 & & \\ & -1 & 0 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} N, \quad \dots, \quad \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} N.$$

Associating these cosets with the involutions $(1\ 2)$, $(2\ 3) \dots (n-1\ n)$ on the set $\{1, \dots, n\}$ we see that $N/T \cong S_n$ the symmetric group on n letters. We note that this coincides with the Weyl group of $\mathfrak{sl}_n(\mathbb{C})$, the Lie algebra of $SL_n(\mathbb{C})$.

We will return to this example later in the section.

3.2 Haar measures

This section defines Haar measures and uses them to construct an inner product on the Lie algebra of a compact Lie group with desirable properties. This inner product will be used throughout the rest of the chapter.

A Lie group G is compact if it is compact as a topological group. Moreover, a real Lie algebra $\mathfrak{g}_0$ is said to be compact if the Lie group $\mathrm{Int}(\mathfrak{g}_0)$ is compact.

Definition 3.2.1 Haar measures

Let G be a compact topological group with topology $\mathcal{T}$ and let $\sigma(\mathcal{T})$ be the σ-algebra generated by $\mathcal{T}$. A measure $\mu : \sigma(\mathcal{T}) \to [0,1]$ is called a normalised Haar measure if it satisfies the following properties.

(i) $\mu(Ex) = \mu(xE) = \mu(E)$ *for all* $x \in G$ *and all* $E \in \sigma(\mathcal{T})$ *(μ is G-invariant),*

(ii) $\mu(U) > 0$ *for all nonempty* $U \in \mathcal{T}$,

(iii) $\mu(G) = 1$,

(iv) *if μ^* is the outer measure of μ then for all $X \subseteq G$ there is some $B \in \sigma(T)$ such that $B \subseteq X$ and $\mu(B) = \mu^*(X)$.*

Normalised Haar measures exist, (see [Ne99], pages 37-40), so we may define a space of Haar integrable functions,

$$\mathcal{L}^2(G) := \left\{ f : G \to \mathbb{C} \;\middle|\; ||f||_2 := \int_G |f(x)|^2 d\mu(x) < \infty \right\}.$$

Let $\sim$ be the equivalence relation on $\mathcal{L}^2(G)$ given by $f \sim g$ if and only if $\mu\{x \in G \,|\, f(x) \neq g(x)\} = 0$. Then $L^2(G) = \mathcal{L}^2(G)/\sim$. However, we still speak of elements of $L^2(G)$ as functions rather than equivalence classes of functions. We require this equivalence relation on $\mathcal{L}^2(G)$ as it will be necessary to define inner products on this space. Without this equivalence there would not be a unique 0 function so no inner product could exist.

We will require an inner product upon which the two maps $\mathrm{Ad}(G)$ and $\mathrm{ad}(G)$ act as orthogonal and anti-symmetric transformations respectively. The next two proposition guarantee the existence of such an inner product.

Definition 3.2.2 Unitary representations

Let G be a topological group and let $\pi : G \to GL(V)$ be a representation of G on a finite dimensional inner product space $(V, \langle\cdot,\cdot\rangle)$. π is said to be unitary if $\langle \pi(g)x, \pi(g)y\rangle = \langle x, y\rangle$ for all $g \in G$.

Proposition 3.2.3 *Let ϕ be a finite dimensional representation of a compact topological group G on V. Then there is a Hermitian inner product $\langle\cdot,\cdot\rangle$ on V such that ϕ is unitary.*

Proof: Let $(\cdot,\cdot)$ be any Hermitian inner product on V (clearly one exists, namely the complex dot product). Define

$$\langle u, v\rangle = \int_G (\phi(x)u, \phi(x)v) d\mu(x).$$

Here, integration is considered over the measure space $(L^2(G), \mu)$, where μ is a normalised Haar measure. $\langle\cdot,\cdot\rangle$ is a Hermitian inner product on V inheriting these properties from $(\cdot,\cdot)$, which is unitary since

$$\langle \phi(y)u, \phi(y)v\rangle = \int_G (\phi(x)\phi(y)u, \phi(x)\phi(y)v) d\mu(x) = \int_G (\phi(xy)u, \phi(xy)v) d\mu(x) = \langle u, v\rangle.$$

Proposition 3.2.4 Existence of a suitable inner product

Let G be a compact Lie group and let $\mathfrak{g}_0$ be its Lie algebra. Then the real vector space $\mathfrak{g}_0$ admits an inner product $\langle\cdot,\cdot\rangle$ that is invariant under $\mathrm{Ad}(G)$, so $\langle \mathrm{Ad}(g)u, \mathrm{Ad}(g)v\rangle = \langle u, v\rangle$, relative to which members of $\mathrm{Ad}(G)$ act by orthogonal transformations, while members of $\mathrm{ad}(G)$ act by anti-symmetric transformations.

Proof: $\mathrm{Ad}(G)$ is a representation of G on $\mathfrak{g}_0$ so by proposition 3.2.3 an inner product exists in which $\mathrm{Ad}(G)$ is a unitary transformation and thus orthogonal. As real inner products are bilinear, this inner product is invariant under $\mathrm{Ad}(G)$.

If we differentiate the equation $\langle \mathrm{Ad}(\exp tX)u, \mathrm{Ad}(\exp tX)v\rangle = \langle u, v\rangle$ at $t = 0$ we see that $\langle (\mathrm{ad}(X))u, v\rangle = -\langle u, (\mathrm{ad}(X))v\rangle$ for all $X \in \mathfrak{g}_0$ so $\mathrm{ad}(X)$ acts anti-symmetrically. □

We note one important corollary to this result.

Corollary 3.2.5 *If G is a compact Lie group with Lie algebra $\mathfrak{g}_0$, then the Killing form of $\mathfrak{g}_0$ is negative semidefinite.*

Proof: Define the inner product $\langle ; \rangle$ as in proposition 3.2.4, also via this proposition we know that $\mathrm{ad}(X)$ is anti-symmetric for all $X \in \mathfrak{g}_0$. Therefore the eigenvalues of $\mathrm{ad}(X)$ are purely imaginary, and thus the eigenvalues of $(\mathrm{ad}(X))^2$ are nonpositive. If B is the Killing form, it follows that
$B(X, X) = \mathrm{Tr}(\mathrm{ad}(X)^2) \leq 0$, as required. □

The following result gives something resembling an inverse to corollary 3.2.5, which will be useful when proving that an abstract Lie algebra is compact.

Proposition 3.2.6 *If the Killing form of a real Lie algebra $\mathfrak{g}_0$ is negative definite, then $\mathfrak{g}_0$ is a compact Lie algebra.*

Proof: If B is negative definite, then it is nondegenerate, so by Cartan's criterion for semisimplicity 1.4.2, $\mathfrak{g}_0$ is semisimple. The uniqueness theorems from chapter 2 prove that $\mathrm{Int}(\mathfrak{g}_0) = (\mathrm{Aut}_{\mathbb{R}}(\mathfrak{g}_0))_0$. Consequently, $\mathrm{Int}(\mathfrak{g}_0)$ is a closed subgroup of $GL(\mathfrak{g})$.

On the other hand, the negation of the Killing form is an inner product on $\mathfrak{g}$ in which every member of $\mathrm{ad}(\mathfrak{g})$ acts as an anti-symmetric transformation. Therefore its corresponding connected Lie group $\mathrm{Int}(\mathfrak{g})$ acts by orthogonal transformations so $\mathrm{Int}(\mathfrak{g}_0)$ is a closed subgroup of the orthogonal group and thus it is compact. □

3.3 Centralisers of Tori

In this section, let G be a compact connected Lie group, let $\mathfrak{g}_0$ be the Lie algebra of G and denote the complexification of $\mathfrak{g}_0$ by $\mathfrak{g}$. We fix an inner product on $\mathfrak{g}_0$ with the properties guaranteed by proposition 3.2.4 and write B for its negation, so B is negative semidefinite, by corollary 3.2.5.

Definition 3.3.1 Tori and maximal tori

Let G be a compact connected Lie group. A subgroup T is called a torus if it is a compact, connected, abelian Lie subgroup of G. A torus T is said to be maximal if it is maximal by inclusion.

Tori are compact, connected, abelian groups which are also differentiable manifolds. It follows from this that they are in fact products of the circle group $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Proposition 3.3.2 *The maximal tori in G are exactly the analytic subgroups corresponding to maximal abelian subalgebras of $\mathfrak{g}_0$.*

Proof: As G is connected we will use the subgroup subalgebra correspondence (1.5.4) in this proof. If T is a maximal torus in G and $\mathfrak{t}_0$ is its Lie algebra, we need to show that $\mathfrak{t}_0$ is maximal abelian. Suppose not, then let $\mathfrak{h}_0$ be an abelian Lie algebra which strictly contains $\mathfrak{t}_0$. The corresponding connected Lie subgroup H will be abelian and will contain T as a strict subset which is a contradiction.

Conversely, suppose $\mathfrak{t}_0$ is a maximal abelian subalgebra in $\mathfrak{g}_0$. The corresponding connected Lie subgroup T must be abelian. If T were not closed then $\overline{T}$ would have a strictly larger abelian Lie algebra, contradicting the maximality of $\mathfrak{t}_0$. Hence T is a maximal torus. □

Consider a compact connected Lie group G.

Let T be a maximal torus in G and let $\mathfrak{t}_0$ be its Lie algebra. We know that $\mathfrak{g}_0 = Z_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ where $Z_{\mathfrak{g}_0}$ is the centre of $\mathfrak{g}_0$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple. Since $\mathfrak{t}_0$ is maximally abelian in $\mathfrak{g}_0$, $t_0 = Z_{\mathfrak{g}_0} \oplus \mathfrak{t}'_0$, where $\mathfrak{t}'_0$ is maximally abelian in $[\mathfrak{g}_0, \mathfrak{g}_0]$.

If we drop the subscript 0's to indicate complexifications we know that $\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}]$ where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple by proposition 1.3.12(iv). Moreover, $t = Z_{\mathfrak{g}} \oplus \mathfrak{t}'$, where $\mathfrak{t}'$ is maximal abelian in $[\mathfrak{g}_0, \mathfrak{g}_0]$. Therefore $\mathfrak{t}'$ is a Cartan subalgebra of the complex semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, so using the root-space decomposition we may write

$$\mathfrak{g} = Z_{\mathfrak{g}} \oplus \mathfrak{t}' \oplus \bigoplus_{a \in \Phi([\mathfrak{g}, \mathfrak{g}], \mathfrak{t}')} [\mathfrak{g}, \mathfrak{g}]_a.$$

Using this, it is clear that $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. If we extend the members a of $\Phi([\mathfrak{g},\mathfrak{g}],\mathfrak{t}')$ to $\mathfrak{t}$ by defining them to be 0 on $Z_{\mathfrak{g}}$, then we may write the root-space decomposition of $\mathfrak{g}$ relative to ad($\mathfrak{t}$) as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{a\in\Phi(\mathfrak{g},\mathfrak{t})} \mathfrak{g}_a \qquad \text{where} \qquad \mathfrak{g}_a = \{X \in \mathfrak{g} \,|\, [H,X] = a(H)X \text{ for all } H \in \mathfrak{t}\}.$$

We say a is a root if $a \neq 0$ and $\mathfrak{g}_a \neq 0$. The nonzero members of $\mathfrak{g}_a$ are called root vectors for the root a and we refer to this equation as the root-space decomposition of $\mathfrak{g}$ relative to $\mathfrak{t}$. In particular the set $\Phi(\mathfrak{g},\mathfrak{t})$ has all the properties expected of an abstract root system, except that it does not necessarily span $\mathfrak{t}^*$.

The presence of the groups G and T give us more information about the root-space decomposition. $\mathrm{Ad}(T)$ acts by orthogonal transformations on $\mathfrak{g}_0$ relative to the inner product chosen. If we extend this inner product on $\mathfrak{g}_0$ to a Hermitian inner product on $\mathfrak{g}$, then $\mathrm{Ad}(T)$ acts as a commuting family of unitary transformations. Using basic representation theory it is clear that $\mathrm{Ad}(T)$ has a simultaneous eigenspace decomposition given by the root-space decomposition. The action of $\mathrm{Ad}(T)$ on the 1-dimensional space $\mathfrak{g}_a$ is a 1-dimensional representation of T so is necessarily of the form $\mathrm{Ad}(t)X = \xi_a(t)X$ for $t \in T$ where $\xi_a : T \to S^1$ is a continuous homomorphism from T into the group of complex numbers of modulus 1. ξ_a is called a multiplicative character. The differential of ξ_a is $a|_{\mathfrak{t}_0}$. Thus $a|_{\mathfrak{t}_0}$ is imaginary valued and the roots are real valued on $i\mathfrak{t}_0$.

Lemma 3.3.3 *Let T be a maximal torus in G, let $\mathfrak{t}$ be the complexification of its corresponding Lie algebra and let Φ be the set of roots. If there is a $H \in \mathfrak{t}$ such that $a(H) \neq 0$ for all $a \in \Phi$ then the centraliser of H in $\mathfrak{g}$ is $\mathfrak{t}$.*

Proof: Let $X \in C_{\mathfrak{g}}(H)$, and let $X = H' + \sum_{a\in\Phi} X_a$ be the decomposition of X according to the root-space decomposition, then

$$0 = [H,X] = 0 + \sum_{a\in\Phi}[H,X_a] = \sum_{a\in\Phi} a(H)X_a$$

Thus $a(H)X_a = 0$ for all a. As $a(H) \neq 0$, $X_a = 0$ so $X = H' \in \mathfrak{t}$. □

Theorem 3.3.4 *If G is a compact connected Lie group then any two maximal abelian subalgebras of $\mathfrak{g}_0$ are conjugates in* $\mathrm{Ad}(G)$.

Proof: Let $\mathfrak{t}_0$ and mft_0' be two maximal abelian subalgebras with T and T' as the corresponding maximal tori. There are only finitely many roots relative to $\mathfrak{t}'$ and therefore the union of their kernels (which are all hyperplanes in $\mathfrak{t}'$) cannot contain $\mathfrak{t}'$. Thus there is some $X \in \mathfrak{t}_0'$ such that $C_{\mathfrak{g}}(X) = \mathfrak{t}'$ by lemma 3.3.3. Similarly, we can find $Y \in \mathfrak{t}_0$ such that $C_{\mathfrak{g}}(X) = \mathfrak{t}'$. Choose a $g_0 \in G$ such that $B(\mathrm{Ad}(g)X, Y)$ is minimised when $g = g_0$. For any $Z \in \mathfrak{g}_0$,

$$r \mapsto B(\mathrm{Ad}(\exp rZ)\mathrm{Ad}(g_0)X, Y)$$

is a smooth function of r which is minimised when $r = 0$. If we differentiate at the point $r = 0$ we see that

$$0 = B((\mathrm{ad}(Z))\mathrm{Ad}(g_0)X, Y) = B([Z, \mathrm{Ad}(g_0)X], Y) = B(Z, [\mathrm{Ad}(g_0)X, Y])$$

Z is arbitrary, so $[\mathrm{Ad}(g_0)X, Y] = 0$. Thus $\mathrm{Ad}(g_0)X \in C_{\mathfrak{g}_0}(T) = \mathfrak{t}_0$. But $\mathfrak{t}_0$ is abelian, so

$$\mathfrak{t}_0 \subseteq C_{\mathfrak{g}_0}(\mathrm{Ad}(g_0)X) = \mathrm{Ad}(g_0)C_{g_0}(X) = \mathrm{Ad}(g_0)\mathfrak{t}_0'$$

Both sides of this equation are abelian, so equality holds as $\mathfrak{t}_o$ is maximal. Thus $\mathfrak{t}_0 = \mathrm{Ad}(\mathfrak{g}_0)\mathfrak{t}_0'$. □

Corollary 3.3.5 *If G is a compact connected Lie group then any two maximal tori are conjugate.*

Proof: Let T and T' be two maximal tori with respective maximal abelian subalgebras $\mathfrak{t}_0$ and $\mathfrak{t}_0'$. Then $\mathfrak{t}_0 = \mathrm{Ad}(g_0)\mathfrak{t}_0'$ for some $g_0 \in G$ by theorem 3.3.4. Hence $T = g_0T'g_0^{-1} = (T')^{g_0}$. □

Theorem 3.3.6 *Let G be a compact connected Lie group and let T be a maximal torus in G. Then every element in G is the conjugate of some element in T. In other words, for each $x \in G$ there are elements $y \in G$ and $t \in T$ such that $x = yty^{-1} = t^y$.*

The proof is extensive and can be found in [Kn02] on pages 255-258. □

Corollary 3.3.7 *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$.*

(i) *Every element of G lies in some maximal torus.*

(ii) *The centre of G, Z_G lies in every maximal torus.*

(iii) *The exponential map from $\mathfrak{g}$ to G is onto.*

Proof:

(i) Let $x \in G$ and let T be a maximal torus of G, then there are elements $y \in G$ and $t \in T$ such that $x = t^y$ by theorem 3.3.6, moreover $x \in T^y$ which is another maximal torus of G.

(ii) Let $z \in Z_G$ and let T be a maximal torus of G. Then there are elements $y \in G$ and $t \in T$ such that $z = yty^{-1}$. Thus $z = zy^{-1}y = y^{-1}zy = t$ so $z \in T$.

(iii) The exponential map is onto for any maximal torus, so by part (i), the exponential map is onto for G. □

Lemma 3.3.8 *Let A be a compact abelian Lie group such that A/A_0 is cyclic, where A_0 is the identity component of A. Then A contains an element a whose powers are dense in A.*

Proof: A_0 is a torus (a product of circle groups) so we can choose some

$$a_0 = (\exp^{2\pi x_i})_{i=1}^{\dim(A_0)} \in A_0$$

such that each x_k is transcendental over $\mathbb{Q}$. Then the powers of a_0 are dense in A_0. Let $N = |A/A_0|$ and let $b \in A$ be a representative of a generating coset in A/A_0. Since $b^N \in A_0$, we can find some $c \in A_0$ with $b^N c^N = a$. Then $a = bc$ is dense in A_0 and contains one element in each coset, thus it is dense in A. □

Theorem 3.3.9 *Let G be a compact connected Lie group and let S be some torus in G. If $g \in C_G(S)$ then there is some torus S' containing both g and S.*

Proof: Let A be the closure of

$$\bigcup_{n=-\infty}^{\infty} g^n S.$$

Then the identity component A_0 of A is a torus. Since A_0 is open in A,

$$\bigcup_{n=-\infty}^{\infty} g^n A_0 \text{ is an open subgroup of } A \text{ containing } \bigcup_{n=-\infty}^{\infty} g^n S.$$

Hence $\bigcup_{n=-\infty}^{\infty} g^n A_0 = A$.

By compactness this open cover of A admits a finite subcover, so there is some non-zero power of g in A_0. Let N denote the smallest such positive power, then A/A_0 is cyclic of order N so we may apply lemma 3.3.8 to find some $a \in A$ whose powers are dense in A. Using corollary 3.3.7(iii), $a = \exp(X)$ for some $X \in \mathfrak{g}_0$. Then the closure of $\{\exp(rX) \,|\, r \in \mathbb{R}\}$ is a torus S' containing A, and thus also containing S and g. □

Corollary 3.3.10 *If G is a compact connected Lie group, the centraliser of a torus is connected. If T is a maximal torus in G then $T = C_G(T)$.*

Proof: By theorem 3.3.9 $C_G(T)$ is the union of the tori which contain T and therefore is a torus, if T is maximal, then $T = C_G(T)$. □

We now recap the ideas presented so far in this section.

Let G be a compact connected Lie group, $\mathfrak{g}_0$ its Lie algebra, T a maximal torus of G and $\mathfrak{t}_0$ its Lie algebra. Let B be the negation of an invariant inner product on G. We indicate the complexification of a Lie algebra by dropping the subscript 0. Let $\Phi(\mathfrak{g}, \mathfrak{t})$ be the set of roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{t}$. The centre $Z_{\mathfrak{g}_0}$ of $\mathfrak{g}_0$ is contained in $\mathfrak{t}_0$, and all roots vanish on $Z_{\mathfrak{g}}$.

We know that the roots are purely imaginary valued on $\mathfrak{t}_0$. We define $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}_0$ which is a real form of $\mathfrak{t}$ in which all the roots are real. We may then regard all roots as lying in $(\mathfrak{t}_{\mathbb{R}})^*$. Then $\Phi(\mathfrak{g}, \mathfrak{t})$ is an abstract root system in the subspace of $\mathfrak{t}_{\mathbb{R}}$ corresponding to the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$.

The negative definite form B on $\mathfrak{t}_0$ then corresponds to a positive definite form on $\mathfrak{t}_{\mathbb{R}}$. Thus given $\lambda \in (\mathfrak{t}_{\mathbb{R}})^*$, let H_λ be the member of $\mathfrak{t}_{\mathbb{R}}$ such that $\lambda(H) = B(H, H_\lambda)$ for all $H \in (\mathfrak{t}_{\mathbb{R}})^*$.

The resulting linear map $\lambda \mapsto H_\lambda$ is a vector space isomorphism from $(\mathfrak{t}_{\mathbb{R}})^*$ to $\mathfrak{t}_{\mathbb{R}}$. Let $(iZ_{\mathfrak{g}_0})^*$ be the subspace of $(\mathfrak{t}_{\mathbb{R}})^*$ corresponding to $iZ_{\mathfrak{g}_0}$. The inner product on $\mathfrak{t}_{\mathbb{R}}$ induces an inner product on $(\mathfrak{t}_{\mathbb{R}})^*$ which we will denote by $\langle \cdot, \cdot \rangle$. With respect to this inner product $\Phi(\mathfrak{g}, \mathfrak{t})$ spans the orthogonal complement of $(iZ_{\mathfrak{g}_0})^*$ and $\Phi(\mathfrak{g}, \mathfrak{t})$ is an abstract reduced root system in this orthogonal complement. Also,

$$\langle \lambda, \mu \rangle = \lambda(H_\mu) = \mu(H_\lambda) = B(H_\lambda, H_\mu)$$

For $a \in \Phi(\mathfrak{g}, \mathfrak{t})$ we define the root reflection as in the semisimple case, namely

$$s_a(\lambda) = \lambda - \frac{2\langle \lambda, a \rangle}{\langle a, a \rangle} a.$$

s_a is the identity transformation on $(iZ_{\mathfrak{g}_0})^*$ and is the usual root reflection in its orthogonal complement. We define the Weyl group $W(\Phi(\mathfrak{g}, \mathfrak{t}))$ as being the group generated by the involutions s_a for $a \in \Phi$. This we think of as the algebraically defined Weyl group.

Definition 3.3.11 **Analytic Weyl group**

Let G be a compact connected Lie group and let T be a maximal torus of G. The analytic Weyl group is given by

$$W(G, T) = N_G(T)/C_G(T) = N_G(T)/T.$$

(The last equality is due to corollary 3.3.10*).*

The group $W(G, T)$ acts by automorphisms of T, hence by invertible linear transformations on $\mathfrak{t}_0$ and the associated spaces $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{t}_0$, $\mathfrak{t}$, $(\mathfrak{t}_{\mathbb{R}})^*$ and $\mathfrak{t}^*$. Moreover, only 1_W acts as the identity. The following theorem explains why both $W(\mathfrak{g}, \mathfrak{h})$ and $W(G, T)$ are referred to as the Weyl group.

Theorem 3.3.12 *Let G be a compact connected Lie group with maximal torus T. The analytic Weyl group $W(G, T)$ and the algebraic Weyl group $W(\Phi(\mathfrak{g}, \mathfrak{t})$ coincide when considered as acting on $(\mathfrak{t}_{\mathbb{R}})^*$.*

Proof: Omitted, see [Kn02] pages 262-264.

Theorem 3.3.12 agrees with the observation made in section 3.1 that the analytic Weyl group of $SL_n(\mathbb{C})$ is S_n and this finally proves that the (algebraic) Weyl group of $\mathfrak{sl}_n(\mathbb{C})$ is also S_n.

3.4 Compact real forms

We aim to show that every complex semisimple Lie algebra has a compact real form. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the corresponding root system and let B be the Killing form of $\mathfrak{g}$.

For each pair $\{a, -a\} \subseteq \Phi$, we fix root vectors $E_a \in \mathfrak{g}_a, E_{-a} \in \mathfrak{g}_{-a}$ such that $B(E_a, E_{-a}) = 1$, by lemma 2.3.6(i) $[E_a, E_{-a}] = H_a$. Let $a, b \in \Phi$, if $a + b \in \Phi$ then we define $C_{a,b}$ such that $[E_a, E_b] = C_{a,b}E_{a+b}$. If $a + b \notin \Phi$ we set $C_{a,b} = 0$. We collect a few preliminary results in the following lemma.

Lemma 3.4.1

(i) $C_{a,b} = -C_{b,a}$.

(ii) *If $a, b, c \in \Phi$ and $a + b + c = 0$ then $C_{a,b} = C_{b,c} = C_{c,a}$.*

(iii) *Let* $a, b, a+b \in \Phi z$*, and let* $\{b + na \mid -p \leq n \leq q\}$ *be the* a *string containing* b*. Then*

$$C_{a,b}C_{-a,-b} = -\frac{1}{2}q(1+p)\langle a, a\rangle.$$

Proof:

(i) The Lie bracket is anti-symmetric.

(ii) By the Jacobi identity, $[[E_a, E_b], E_c] + [[E_b, E_c], E_a] + [[E_c, E_a], E_b] = 0$.

Thus $C_{a,b}[E_{-c}, E_c] + C_{b,c}[E_{-a}, E_a] + C_{c,a}[E_{-b}, E_b] = C_{a,b}H_c + C_{b,c}H_a + C_{c,a}H_b = 0$.

Substituting $H_c = -H_a - H_b$ and using the linear independence of the set $\{H_a, H_b\}$ we get the desired result.

(iii) By corollary 2.3.12 $[E_{-a}, [E_a, E_b]] = \frac{1}{2}q(1+p)\langle a, a\rangle B(E_a, E_{-a})E_b$

$[E_{-a}, [E_a, E_b]] = C_{-a,a+b}C_{a,b}E_b$ and $B(E_a, E_{-a}) = 1$, so $C_{-a,a+b}C_{a,b} = \frac{1}{2}q(1+p)\langle a, a\rangle$

$(-a) + (a+b) + (-b) = 0$ so by parts (i) and (ii) $C_{-a,a+b} = C_{-b,-a} = -C_{-a,-b}$. □

Theorem 3.4.2 *Let* $\mathfrak{g}$ *be a complex semisimple Lie algebra and let* $\mathfrak{h}$ *be a Cartan subalgebra of* $\mathfrak{g}$ *and let* Φ *be the set of roots. Given* $a \in \Phi$*, there is a root vector* $X_a \in \mathfrak{g}_a$ *such that for all* $a, b \in \Phi$*,*

$$[X_a, X_{-a}] = H_a \text{ and } [X_a, X_b] = \begin{cases} N_{a,b} & \text{whenever } a+b \in \Phi \\ 0 & \text{otherwise} \end{cases}$$

Moreover, $N_{a,b} = -N_{-a,-b}$*. Moreover, for any such choice of system* $\{X_a\}$ *of root vectors,*

$$N_{a,b}^2 = \frac{1}{2}q(1+p)\langle a, a\rangle, \text{ where } \{b + na \mid -p \leq n \leq q\} \text{ is the } a \text{ string containing } b.$$

Proof: The transpose of the linear map $\psi : \mathfrak{h} \to \mathfrak{h}$ given by $\psi(h) = -h$ carries Φ to Φ and thus extends to an automorphism $\widetilde{\psi}$ of $\mathfrak{g}$ by the isomorphism theorem 2.7.11. $\widetilde{\psi}(E_a) \in \mathfrak{g}_{-a}$ so there exists a constant c_{-a} such that $\widetilde{\psi}(E_a) = c_{-a}E_{-a}$ (as the space $\mathfrak{g}_a$ is 1 dimensional).

By proposition 1.3.16, the Killing form is invariant under automorphisms so $B(\widetilde{\psi}X, \widetilde{\psi}Y) = B(X, Y)$ for all $X, Y \in \mathfrak{g}$.

Setting $X = E_a$ and $Y = E_{-a}$ we see that

$$c_{-a}c_a = c_{-a}c_aB(E_{-a}, E_a) = B(\widetilde{\phi}E_a, \widetilde{\phi}E_{-a}) = B(E_a, E_{-a}) = 1.$$

Hence we may choose d_a for each $a \in \Phi$ such that $d_ad_{-a} = 1$ and $(d_a)^2 = -c_a$, for example by writing $c_a = re^{i\theta}$, so that $c_{-a} = r^{-1}e^{-i\theta}$; then taking

$$d_a = r^{\frac{1}{2}}ie^{i\frac{\theta}{2}} \text{ and } d_a = r^{-\frac{1}{2}}ie^{-i\frac{\theta}{2}}.$$

These satisfy both the conditions we required. Define $X_a = d_aE_a$ and notice that

$$[X_a, X_{-a}] = d_ad_{-a}[E_a, E_{-a}] = H_a \qquad \text{and}$$

$$\widetilde{\psi}(X_a) = d_a\widetilde{\psi}(E_a) = d_ac_{-a}E_{-a} = d_{-a}^{-1}c_{-a}E_{-a} = -d_{-a}E_{-a} = -X_{-a}$$

Define $N_{a,b}$ such that $[X_a, X_b] = N_{a,b}X_{a+b}$ whenever $a+b \in \Phi$ and $N_{a,b} = 0$ otherwise.

Then $-N_{a,b}X_{-a-b} = \widetilde{\psi}(N_{a,b}X_{a+b}) = \widetilde{\psi}[X_a, X_b] = [\widetilde{\psi}X_a, \widetilde{\psi}X_b] = [-X_{-a}, -X_{-b}] = N_{-a,-b}X_{-a-b}$, so it follows that $N_{a,b} = -N_{-a,-b}$.

The formula for $N_{a,b}^2$ follows by repeating the argument in lemma 3.4.1(iii), replacing all the root vectors E_a by X_a. □

Definition 3.4.3 Split real forms

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h}$ be some Cartan subalgebra of $\mathfrak{g}$. Define

$$\mathfrak{h}_0 = \{H \in \mathfrak{h} \,|\, a(H) \in \mathbb{R} \text{ for all } a \in \Phi\}$$

A real form of $\mathfrak{g}$ which contains $\mathfrak{h}_0$ for some Cartan subalgebra $\mathfrak{h}$ is called a split real form of $\mathfrak{g}$.

Proposition 3.4.4 *Any complex semisimple Lie algebra contains a split real form.*

Proof: Write $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{a\in\Phi} \mathbb{R}X_a$. The formula $N_{a,b}^2 = \frac{1}{2}q(1-p)\langle a, a\rangle > 0$ shows that $N_{a,b} \in \mathbb{R}$. It is clear that $g^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$, when considered as real vector spaces so $\mathfrak{g}_0$ is a split real form of $\mathfrak{g}$. □

Definition 3.4.5 Compact real forms

A compact real form of a complex semisimple Lie algebra $\mathfrak{g}$ is a real form of $\mathfrak{g}$ which is also a compact Lie algebra.

Theorem 3.4.6 *Every complex semisimple Lie algebra admits a compact real form $\mathfrak{u}_0$.*

Proof: The proof shows that the following choice of $\mathfrak{u}_0$ is suitable.

$$\mathfrak{u}_0 = \sum_{a\in\Phi} \left(\mathbb{R}(iH_a) + \mathbb{R}(X_a - X_{-a}) + \mathbb{R}i(X_a - X_{-a})\right).$$

This follows by showing that $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u}_0 + i\mathfrak{u}_0$ as real vector spaces, $\mathfrak{u}_0$ is compact and closed under the bracket operation. To show compactness we deduce that the restriction of the Killing form $B|_{\mathfrak{u}_0\times\mathfrak{u}_0}$ is negative definite. The full details can be found in [Kn02], pages 353-354. □

Using the formula given in proposition 3.4.4 it is clear that $\mathfrak{sl}_n(\mathbb{R})$ is a split real form of $\mathfrak{sl}_n(\mathbb{C})$. It is less obvious that the formula given by theorem 3.4.6 defines $\mathfrak{su}_n(\mathbb{C})$ as a compact real form of $\mathfrak{sl}_n(\mathbb{C})$.

3.5 Cartan decomposition

Definition 3.5.1 Cartan involution

Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and let $\theta : \mathfrak{g}_0 \to \mathfrak{g}_0$ be an involution (an automorphism with $\theta^2 = 1$). Let B be the Killing form and define the symmetric bilinear form B_θ by $B_\theta(X, Y) = -B(X, \theta Y)$. If B_θ is positive definite then θ is called a Cartan involution.

Proposition 3.5.2 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{u}_0$ be a compact real form of $\mathfrak{g}$ (guaranteed by theorem 3.4.6). Let τ be the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{u}_0$. If $\mathfrak{g}$ is regarded as the real Lie algebra $\mathfrak{g}^{\mathbb{R}}$ then τ is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$.*

Proof: τ is certainly an involution. Moreover, the Killing forms $B_{\mathfrak{g}}$ of $\mathfrak{g}$ and $B_{\mathfrak{g}^{\mathbb{R}}}$ of $\mathfrak{g}^{\mathbb{R}}$ are related by the formula $B_{\mathfrak{g}^{\mathbb{R}}}(Z_1, Z_2) = 2\,\mathrm{Re}(B_{\mathfrak{g}}(Z_1, Z_2))$,

Write $Z = X + iY$ where $X, Y \in \mathfrak{u}_0$, then

$B_{\mathfrak{g}}(Z, \tau Z) = B_{\mathfrak{g}}(X + iY, X - iY) = B_{\mathfrak{g}}(X, X) + B_{\mathfrak{g}}(Y, Y) = B_{\mathfrak{u}_0}(X, X) + B_{\mathfrak{u}_0}(Y, Y) < 0$ unless $Z = 0$.

Hence, $(B_{\mathfrak{g}^{\mathbb{R}}})_\tau(Z_1, Z_2) = -B_{\mathfrak{g}^{\mathbb{R}}}(Z_1, \tau Z_2) = -2\,\mathrm{Re}(B_{\mathfrak{g}}(Z_1, \tau Z_2))$ is positive definite on $\mathfrak{g}^{\mathbb{R}}$ and thus τ is a Cartan involution. □

Lemma 3.5.3 *Let $\mathfrak{g}_0$ be a real finite-dimensional Lie algebra and let ρ be an automorphism of $\mathfrak{g}_0$ that is diagonalisable with positive eigenvalues $d_1, \ldots, d_m$ and corresponding eigenspaces $(\mathfrak{g}_0)_{d_j}$.*

For $r \in \mathbb{R}$ define ρ^r to be the linear transformation that is given by multiplication by $(d_j)^r$ on $(\mathfrak{g}_0)_{d_j}$. Then $\{\rho^r\}$ is a one-parameter subgroup of $\mathrm{Aut}(\mathfrak{g}_0)$. If $\mathfrak{g}_0$ is semisimple then $\{\rho^r\}$ lies in $\mathrm{Int}(\mathfrak{g}_0)$.

Proof: Let $X \in (\mathfrak{g}_0)_{d_i}$ and let $Y \in (\mathfrak{g}_0)_{d_j}$, then $\rho[X,Y] = [\rho X, \rho Y] = d_i d_j [X,Y] \in (\mathfrak{g}_0)_{d_i d_j}$, since ρ is an automorphism.

Therefore $\rho^r[X,Y] = (d_i d_j)^r [X,Y] = [d_i^r X, d_j^r Y] = [\rho^r X, \rho^r Y]$, so ρ^r is an automorphism. Thus $\{\rho^r\}$ is a one-parameter subgroup of $\mathrm{Aut}(\mathfrak{g}_0)$ and hence lies in the identity component $\mathrm{Aut}(\mathfrak{g}_0)_0$. If $\mathfrak{g}_0$ is semisimple then $\mathrm{Aut}(\mathfrak{g}_0)_0 = \mathrm{Int}(\mathfrak{g}_0)$. □

Theorem 3.5.4 *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra, let θ be a Cartan involution and let σ be any involution. Then there is a $\psi \in \mathrm{Int}(\mathfrak{g}_0)$ such that $\psi\theta\psi^{-1}$ commutes with σ.*

The proof shows that setting $\psi = \rho^{\frac{1}{4}}$ suffices where $\rho = (\sigma\theta)^2$ and lemma 3.5.3 is used to show that $\{\rho^r\}$ is a one parameter subgroup of $\mathrm{Int}(\mathfrak{g}_0)$. The full proof can be found in [Kn02] on page 357. □

Corollary 3.5.5 *If $\mathfrak{g}_0$ is a real semisimple Lie algebra then $\mathfrak{g}_0$ has a Cartan involution.*

Proof: Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$ and use theorem 3.4.6 to choose a compact real form $\mathfrak{u}_0$ of $\mathfrak{g}$. Let σ and τ be the conjugations of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ and $\mathfrak{u}_0$ respectively. Notice σ and τ are both involutions on $\mathfrak{g}^{\mathbb{R}}$. By proposition 3.5.2, τ is a Cartan involution and by theorem 3.5.4 there is some $\psi \in \mathrm{Int}(\mathfrak{g}^{\mathbb{R}})$ such that $\psi\tau\psi^{-1}$ commutes with σ. Set $\theta = (\psi\tau\psi^{-1})|_{\mathfrak{g}_0}$, then θ is the required Cartan involution.

To see this notice that $\mathfrak{g}_0 = \{X \in \mathfrak{g} \,|\, \sigma X = X\}$ and that $\sigma(\psi\tau\psi^{-1}X) = \psi\tau\psi^{-1}(\sigma X) = \psi\tau\psi^{-1}X$.

Moreover, $B_\theta(X,Y) = -B_{\mathfrak{g}_0}(X, \theta Y) = -B_{\mathfrak{g}}(X, \psi\tau\psi^{-1}Y) = \frac{1}{2}(B_{\mathfrak{g}^{\mathbb{R}}})_{\psi\tau\psi^{-1}}(X,Y)$.

Thus B_θ is positive definite on $\mathfrak{g}_0$ and θ is a Cartan involution. □

Corollary 3.5.6 *If $\mathfrak{g}_0$ is a real semisimple Lie algebra then any two Cartan involutions of $\mathfrak{g}_0$ are conjugate via* $\mathrm{Int}(\mathfrak{g}_0)$.

Proof: Let θ and θ' be two Cartan involutions. Take $\sigma = \theta'$ as in theorem 3.5.4 then there is some $\psi \in \mathrm{Int}(\mathfrak{g}_0)$ such that $\psi\theta\psi^{-1}$ commutes with θ'. $\psi\theta\psi^{-1}$ is another Cartan involution of $\mathfrak{g}_0$ so we may assume that θ and θ' commute.

Since θ and θ' commute, they have a compatible eigenspace decomposition into ± 1 eigenspaces. By symmetry it suffices to show that no X lies in the $+1$ eigenspace of θ and the -1 eigenspace of θ'.

Suppose for a contradiction that $\theta X = X$, and $\theta' X = -X$. Then,

$$-B(X,X) = -B(X, \theta X) = B_\theta(X,X) > 0 \text{ and } B(X,X) = -B(X, \theta' X) = B_{\theta'}(X,X) > 0$$

which is a contradiction. □

Corollary 3.5.7 *If $\mathfrak{g}$ is a complex semisimple Lie algebra then any two compact real forms of $\mathfrak{g}$ are conjugate via* $\mathrm{Int}(\mathfrak{g})$.

Proof: Each compact real form defines a conjugation on $\mathfrak{g}$ which is a Cartan involution by proposition 3.5.2. Applying 3.5.6 to $\mathfrak{g}^{\mathbb{R}}$ these two conjugations are conjugate by an element of $\mathrm{Int}(\mathfrak{g}^{\mathbb{R}}) = \mathrm{Int}(\mathfrak{g})$. □

Corollary 3.5.8 *If $A = (A_{ij})$ is an abstract Cartan matrix, then there is a unique (up to isomorphism) compact semisimple Lie algebra $\mathfrak{g}_0$ whose complexification $\mathfrak{g}$ has a root system with A as its Cartan matrix.*

Proof: The existence of such a $\mathfrak{g}$ is given by theorem 2.7.13 and the uniqueness by corollary 2.7.12. Passing from $\mathfrak{g}$ to $\mathfrak{g}_0$ is enabled by theorem 3.4.6 and corollary 3.5.7. □

Corollary 3.5.9 *If $\mathfrak{g}$ is a complex semisimple Lie algebra then the only Cartan involutions of $\mathfrak{g}^{\mathbb{R}}$ are the conjugations with respect to the compact real forms of $\mathfrak{g}$.*

Proof: By theorem 3.4.6 and proposition 3.5.2 there is a Cartan involution of $\mathfrak{g}^{\mathbb{R}}$ given by conjugation with respect to some compact real form of $\mathfrak{g}$. Any other Cartan involution is conjugate to this one by corollary 3.5.6 and hence is also the conjugation with respect to a compact real form of $\mathfrak{g}$. □

A Cartan involution θ of $\mathfrak{g}_0$ yields an eigenspace decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ of $\mathfrak{g}_0$ into 1 and -1 eigenspaces respectively. Moreover, since θ is an involution:

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0. \qquad (\dagger)$$

Thus it follows that $\mathfrak{k}_0, \mathfrak{p}_0$ are orthogonal with respect to $B_{\mathfrak{g}_0}$ and also with respect to B_θ. In fact, if $X \in \mathfrak{k}_0, Y \in \mathfrak{p}_0$ then ad X ad Y carries $\mathfrak{k}_0$ to $\mathfrak{p}_0$ and $\mathfrak{p}_0$ to $\mathfrak{k}_0$. Thus it must have trace 0 so $B_{\mathfrak{g}_0}(X, Y) = 0$ and also $B_\theta(X, Y) = 0$ since $\theta Y = -Y$.

B_θ is positive definite, so the eigenspaces $\mathfrak{k}_0, \mathfrak{p}_0$ have the property that:

$$B_{\mathfrak{g}_0} \text{ is } \begin{cases} \text{negative definite on } \mathfrak{k}_0 \\ \text{positive definite on } \mathfrak{p}_0 \end{cases} \qquad (\ddagger)$$

A decomposition of $\mathfrak{g}_0$ into two subalgebras which satisfies ($\dagger$) and ($\ddagger$) is called a Cartan decomposition of $\mathfrak{g}_0$.

Conversely, a Cartan decomposition determines a Cartan involution θ by setting

$$\theta(x + y) = x - y \text{ where } z = x + y \text{ is the decomposition of } z \text{ in } \mathfrak{k}_0 + i\mathfrak{p}_0.$$

θ respects brackets and B_θ is positive definite as required.

If $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a Cartan decomposition of $\mathfrak{g}_0$ then $\mathfrak{k}_0 \oplus i\mathfrak{p}_0$ is a compact real form of $\mathfrak{g}_0$, moreover if $\mathfrak{h}_0$ and $\mathfrak{q}_0$ are the +1 and -1 eigenspaces of an involution σ, then σ is a Cartan subalgebra of $\mathfrak{g}_0$ only if $\mathfrak{h}_0 \oplus i\mathfrak{q}_0$ is a compact real form of $\mathfrak{g}$.

So the most general Cartan decomposition of $\mathfrak{g}^\mathbb{R}$ is $\mathfrak{g}^\mathbb{R} = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$ where $\mathfrak{u}_0$ is a compact real form of $\mathfrak{g}$.

Corollaries 3.5.5 and 3.5.6 show that an arbitrary real semisimple Lie algebra $\mathfrak{g}_0$ has a unique Cartan decomposition up to conjugacy by $\mathrm{Int}(\mathfrak{g}_0)$.

Lemma 3.5.10 *If $\mathfrak{g}_0$ is a real semisimple Lie algebra and θ is a Cartan involution, then*

$$(\mathrm{ad}(X))^* = -\mathrm{ad}(\theta X) \text{ for all } X \in \mathfrak{g}_0,$$

where the adjoint operator $(\,\cdot\,)^*$ is defined relative to the inner product B_θ.

Proof:

$$\begin{aligned} B_\theta((\mathrm{ad}(\theta(X)Y), Z) &= -B([\theta(X), Y], \theta(Z)) = B(Y, [\theta(X), \theta(Z)]) \\ &= B(Y, \theta[X, Z]) = -B_\theta(Y, (\mathrm{ad}(X))Z) \\ &= -B_\theta((\mathrm{ad}(X))^* Y, Z). \end{aligned}$$

The linearity of B_θ in the first variable gives the result. □

Proposition 3.5.11 *If $\mathfrak{g}_0$ is a real semisimple Lie algebra, then $\mathfrak{g}_0$ is isomorphic to a Lie algebra of real matrices that is closed under transposition. For any Cartan involution θ of $\mathfrak{g}_0$ this isomorphism may be chosen so that θ is carried to the negation of the transpose.*

Proof: Let θ be a Cartan involution of $\mathfrak{g}_0$ and define the inner product B_θ on $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is semisimple, the natural association $X \mapsto \mathrm{ad}(X)$ shows $\mathfrak{g}_0 \cong \mathrm{ad}(\mathfrak{g}_0)$. Therefore, the matrices of $\mathrm{ad}(\mathfrak{g}_0)$ in an orthonormal basis relative to B_θ will be the required Lie algebra of matrices. The closure of $\mathrm{ad}(\mathfrak{g}_0)$ under the adjoint map follows from lemma 3.5.10 and the fact that $\mathfrak{g}_0$ is closed under θ. □

Corollary 3.5.12 *If $\mathfrak{g}_0$ is a real semisimple Lie algebra and θ is a Cartan involution then any θ stable subalgebra $\mathfrak{s}_0$ of $\mathfrak{g}_0$ is reductive.*

Proof: By proposition 3.5.11 $\mathfrak{g}_0$ may be regarded as a real Lie algebra of real matrices which is closed under transposition and where θ is mapped to the negation of the transpose. Then $\mathfrak{s}_0$ is a Lie subalgebra of matrices closed under transpose and so the result follows from the fact that any such real Lie algebra of matrices is reductive. □

We can now lift these results from the Lie algebra to the Lie group.

Proposition 3.5.13 *If G is a semisimple Lie group and Z is its centre then G/Z has trivial centre.*

Remark: We know that Z is discrete as it is a closed subgroup of G whose Lie algebra is 0, since G is semisimple.

Proof: Let $\mathfrak{g}_0$ be the Lie algebra of G. Given $x \in G$, $\mathrm{Ad}(x)$ represents conjugation by x and $\mathrm{Ad}(x) = 1$ if and only if $x \in Z$. Thus $G/Z \cong \mathrm{Ad}(G)$. If $g \in \mathrm{Ad}(G)$ is central then $g\mathrm{Ad}(x) = \mathrm{Ad}(x)g$ for all $x \in G$.

Differentiating gives $g(\mathrm{ad}(X)) = (\mathrm{ad}(X))g$ for all $X \in \mathfrak{g}_0$. Applying both sides of this equation to some $Y \in \mathfrak{g}_0$ tells us that $g([X,Y]) = [X, gY]$. Thus replacing Y by $g^{-1}Y$ we see that $[gX, Y] = [X, Y]$. Interchanging X and Y gives $[X, gY] = [X, Y]$ and hence $g([X,Y]) = [X,Y]$.

$\mathfrak{g}_0$ is semisimple, so $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{g}_0$. Therefore g is the identity map on $\mathfrak{g}_0$. Thus $\mathrm{Ad}(G)$ has trivial centre. □

Theorem 3.5.14 *Let G be a semisimple Lie group, let θ be a Cartan involution of its Lie algebra $\mathfrak{g}_0$ and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. Let K be the connected Lie subgroup of g with Lie algebra $\mathfrak{k}_0$. Then*

(i) *there is a Lie group involution Θ of G with differential θ,*

(ii) $\{g \in G \,|\, \Theta(g) = g\} = K$,

(iii) *the mapping $K \times \mathfrak{p}_0 \to G$ given by $(k, X) \mapsto k \exp X$ is a diffeomorphism,*

(iv) *K is closed,*

(v) $Z = Z(G) \leq K$,

(vi) *K is compact if and only if Z is finite,*

(vii) *when Z is finite, K is a maximal compact subgroup of G.*

The automorphism Θ guaranteed by this theorem is called the global Cartan involution and the decomposition in (iii) is called the global Cartan decomposition.

Proof: Omitted, see [Kn02] pages 362-367.

3.6 Iwasawa decomposition

As with the Cartan decomposition, we begin with a decomposition of Lie algebras before lifting the result to Lie groups. For this section it is wise to change notation briefly.

G will denote a semisimple Lie group and its Lie algebra will be denoted by $\mathfrak{g}$ (not by $\mathfrak{g}_0$ as previously). If θ is a Cartan involution of $\mathfrak{g}$ we will write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ as the corresponding Cartan decomposition and K for the connected Lie subgroup of G with Lie algebra $\mathfrak{k}$. B will be some nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$ such that $B(\theta X, \theta Y) = B(X, Y)$ for all $X, Y \in \mathfrak{g}$ and B_θ (defined by $B_\theta(X, Y) = B(X, \theta Y)$) is positive definite.

It follows from this that B is negative definite on the compact real form $\mathfrak{k} \oplus i\mathfrak{p}$. Thus B is negative definite on a maximal abelian subspace of $\mathfrak{k} \oplus i\mathfrak{p}$ and hence given any Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, B is positive definite on the subspace where all roots are real-valued. B_θ is an inner product on $\mathfrak{g}$ which we use to define orthogonality and adjoints.

Definition 3.6.1 Restricted root-space decomposition

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, such a thing exists because $\mathfrak{p}$ is finite-dimensional. Since $\mathrm{ad}(X)^* = -\mathrm{ad}(\theta X)$ by lemma 3.5.10, the set $\{\mathrm{ad}(H) \,|\, H \in \mathfrak{a}\}$ is a commuting family of self-adjoint transformations of $\mathfrak{g}$. Then $\mathfrak{g}$ is the orthogonal direct sum of simultaneous eigenspaces with all the corresponding eigenvalues being real. Fixing such an eigenspace with eigenvalue λ_H under $\mathrm{ad}(H)$, then the equation $\mathrm{ad}(H)X = \lambda_H X$ shows λ_H is linear in H. Hence the simultaneous eigenvalues are members of the dual space $\mathfrak{a}^*$. Given $\lambda \in \mathfrak{a}^*$ we write

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \,|\, (\mathrm{ad}(H))X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\mathfrak{g}_\lambda \neq 0$ and $\lambda \neq 0$ we call λ a restricted root of $\mathfrak{g}$ or a root of $(\mathfrak{g}, \mathfrak{a})$. The set of restricted roots is denoted by Σ. Any non-zero $\mathfrak{g}_\lambda$ is called a restricted root-space and its members are called restricted root vectors of the restricted root λ.

Proposition 3.6.2 *The restricted roots and restricted root-spaces have the following properties:*

(i) $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda$,

(ii) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subseteq \mathfrak{g}_{\lambda+\mu}$,

(iii) $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$ *and hence* $\lambda \in \Sigma$ *if and only if* $-\lambda \in \Sigma$,

(iv) $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ *where* $\mathfrak{a}$ *and* $\mathfrak{m}$ *are orthogonal and* $\mathfrak{m} = C_\mathfrak{k}(\mathfrak{a})$.

Remark: The decomposition given by (i) is called the restricted root-space decomposition of $\mathfrak{g}$.

Proof:

(i) Follows by construction.

(ii) Holds by the Jacobi identity.

(iii) Consider $X \in \mathfrak{g}_\lambda$, then $[X, \theta X] = \theta[\theta H, X] = -\theta[H, X] = -\lambda(H)\theta X$.

(iv) By part (iii), $\theta\mathfrak{g}_0 = \mathfrak{g}_0$, hence $\mathfrak{g}_0 = (\mathfrak{k} \cap \mathfrak{g}_0) \oplus (\mathfrak{p} \cap \mathfrak{g}_0)$. Since $\mathfrak{a} \subseteq \mathfrak{p} \cap \mathfrak{g}_0$ and $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$, $\mathfrak{a} = \mathfrak{p} \cap \mathfrak{g}_0$. Also $\mathfrak{k} \cap \mathfrak{g}_0 = C_\mathfrak{k}(\mathfrak{a})$. □

Let Σ^+ be the set of positive roots with respect to some suitable notion of positivity (for example a lexicographical ordering). Define $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$. By proposition 3.6.2(ii), $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$.

Proposition 3.6.3 Iwasawa decomposition of Lie algebras

With the notation given previously, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Moreover, $\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent and $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable Lie subalgebra of $\mathfrak{g}$. Finally, $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

Proof: We know that $\mathfrak{a}$ is abelian and $\mathfrak{n}$ is nilpotent. Since $[\mathfrak{a}, \mathfrak{g}_\lambda] = \mathfrak{g}_\lambda$ for each $\lambda \neq 0$, we see that $[\mathfrak{a}, \mathfrak{n}] = \mathfrak{n}$ and that $\mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra with $[\mathfrak{a} \oplus \mathfrak{n}, \mathfrak{a} \oplus \mathfrak{n}] = \mathfrak{n}$.

To prove that $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is a direct sum it suffices to show that $\mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n}) = 0$. Let $X \in \mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n})$. Then $\theta(X) = X \in \mathfrak{a} \oplus \theta(\mathfrak{n})$. Since $\mathfrak{a} \oplus \mathfrak{n} \oplus \theta(\mathfrak{n})$ is a direct sum by proposition 3.6.2 parts (i) and (iii), $X \in \mathfrak{a}$, but then $X \in \mathfrak{k} \cap \mathfrak{p} = 0$.

Finally, $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \supseteq \mathfrak{g}$ because we can write $X \in \mathfrak{g}$ using some $H \in \mathfrak{a}$ some $X_0 \in \mathfrak{n}$ and elements $X_\lambda \in \mathfrak{g}_\lambda$ by

$$X = H + X_0 + \sum_{\lambda \in \Sigma} X_\lambda = (X_0 + \sum_{\lambda \in \Sigma^+} (X_{-\lambda} + \theta X_{-\lambda})) + H + (\sum \lambda \in \Sigma^+ (X_\lambda - \theta X_{-\lambda}))$$

which lies in $\mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. □

In order to prove the Iwasawa decomposition of Lie groups we first prove two lemmas.

Lemma 3.6.4 *Let H be an connected Lie group with Lie algebra $\mathfrak{h}$ and suppose that $\mathfrak{h}$ is a vector space direct sum of Lie subalgebras $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{t}$. If S and T denote the connected subgroups of G with Lie algebras $\mathfrak{s}$ and $\mathfrak{t}$ respectively, then the multiplication map $\phi(s, t) = st$ of $S \times T$ into H is everywhere regular, i.e.* $\det(d\phi) \neq 0$.

Proof: Let $X \in \mathfrak{s}, Y \in \mathfrak{t}$. Then

$$\phi(s_0 \exp(rX), t_0) = s_0 \exp(rX) t_0 = s_0 t_0 \exp(\mathrm{Ad}(t_0^{-1}) rX) \text{ and } \phi(s_0, t_0 \exp(rY)) = s_0 t_0 \exp(rY).$$

Therefore, $d\phi(X) = \mathrm{Ad}(t_0^{-1})X$ and $d\phi(Y) = Y$.

So $d\phi$ must be block triangular in matrix form, thus

$$\det d\phi = \frac{\det \mathrm{Ad}_\mathfrak{h}(t_0^{-1})}{\det \mathrm{Ad}_\mathfrak{t}(t_0^{-1})} = \frac{\det \mathrm{Ad}_\mathfrak{t}(t_0)}{\det \mathrm{Ad}_\mathfrak{h}(t_0)} \neq 0.$$

Lemma 3.6.5 *There is a basis $\{X_i\}$ of $\mathfrak{g}$ such that the matrices representing* $\mathrm{ad}(\mathfrak{g})$ *have the following properties.*

(i) *the matrices of* $\mathrm{ad}(\mathfrak{k})$ *are anti-symmetric,*

(ii) *the matrices of* $\mathrm{ad}(\mathfrak{a})$ *are diagonal with real entries,*

(iii) *the matrices of* $\mathrm{ad}(\mathfrak{n})$ *are strictly upper triangular.*

Proof: Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{g}$ compatible with the restricted root-space decomposition and with the property that if $X_i \in \mathfrak{g}_{\lambda_i}$, $X_j \in \mathfrak{g}_{\lambda_j}$ with $i < j$ then $\lambda_i \geq \lambda_j$.

For $X \in \mathfrak{k}$, $\mathrm{ad}(X)^* = -\mathrm{ad}(\theta X) = -\mathrm{ad}(X)$ by lemma 3.5.10. Since each X_i is a restricted root vector or is in $\mathfrak{g}_0$, the matrices of $\mathrm{ad}(\mathfrak{a})$ are necessarily diagonal with real entries and the matrices of $\mathrm{ad}(\mathfrak{n})$ are strictly upper triangular by proposition 3.6.2(ii). □

Theorem 3.6.6 Iwasawa decomposition for Lie groups

Let G be a semisimple Lie group, let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G. Let A and N be the connected Lie subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ respectively.
Then the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism. Moreover A and N are simply connected.

Proof: Let $\overline{G} = \mathrm{Ad}(G)$ which can be regarded as the closed subgroup $\mathrm{Aut}(\mathfrak{g})_0$ of $GL(\mathfrak{g})$. We prove the theorem for $\overline{G}$ and then lift the result to G.

Impose the inner product B_θ on $\mathfrak{g}$. We write matrices for elements of $\overline{G}$ and $\mathrm{ad}(\mathfrak{g})$ relative to the basis given by lemma 3.6.5. Let $\overline{K} = \mathrm{Ad}_{\mathfrak{g}}(K)$, $\overline{A} = \mathrm{Ad}_{\mathfrak{g}}(A)$ and $\overline{N} = \mathrm{Ad}_{\mathfrak{g}}(N)$. Lemma 3.6.5 tells us what matrices in these sets look like. Namely, matrices in $\overline{K}$ are rotation matrices, $\overline{A}$ contains diagonal matrices with positive entries and $\overline{N}$ contains upper unitriangular matrices. $\overline{K}$ is compact by proposition 3.5.13 and theorem 3.5.14(vi).

The subgroup of $GL(\mathfrak{g})$ consisting of diagonal matrices with positive entries is abelian and simply connected and $\overline{A}$ is an connected Lie subgroup of it. Therefore $\overline{A}$ is closed in $GL(\mathfrak{g})$ and hence in $\overline{G}$, similarly $\overline{N}$ is closed in $\overline{G}$.

The map $\overline{A} \times \overline{N} \to GL(\mathfrak{g})$ given by $(\overline{a}, \overline{n}) \mapsto \overline{an}$ is injective since we can recover $\overline{a}$ from the diagonal entries of $\overline{an}$. Clearly, this function is onto from $\overline{A} \times \overline{N} \to \overline{AN}$. Since $\overline{A}, \overline{N}$ are both closed a simple limit argument shows that $\overline{AN}$ is closed and has the Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$. Thus by lemma 3.6.4, $\overline{A} \times \overline{N} \to \overline{AN}$ is a diffeomorphism.

K is compact and thus the image of the map $\overline{K} \times \overline{A} \times \overline{N} \to \overline{K} \times \overline{AN} \to \overline{G}$ is the product of a compact set with a closed set and thus it is closed. Also, the image is open since the map is everywhere regular by lemma 3.6.4. By dimensional considerations from the Iwasawa decomposition of $\mathfrak{g}$ it follows that the image is all of $\overline{G}$.

Thus the multiplication map is smooth, regular and onto. Finally, $\overline{K} \cap (\overline{A} \times \overline{N}) = \{1\}$ since a rotation matrix with positive eigenvalues is always 1. Hence the multiplication map $\overline{K} \times \overline{A} \times \overline{N} \to \overline{G}$ is a diffeomorphism.

Let $e : G \to \overline{G} = \mathrm{Ad}(G)$ be the natural (smooth) homomorphism $e(g) = \mathrm{Ad}(g)$. We may define an inverse of e locally, so we may write this map locally as

$$(k, a, n) \mapsto (e(k), e(a), e(n)) \mapsto e(k)e(a)e(n) = e(kan) \mapsto kan$$

and therefore the multiplication map is smooth and everywhere regular. Since $\overline{A}$ and $\overline{N}$ are simply connected $e|_A$ and $e|_N$ are injective covering maps to $\overline{A}$ and $\overline{N}$ respectively, from this we can deduce that A and N are simply connected.

To show the multiplication map is onto, let $g \in G$ and write $e(g) = \overline{k}\overline{an}$. Set

$$a = (e_A)^{-1}(\overline{a}) \in A, \text{ and } n = (e_N)^{-1}(\overline{n}) \in N$$

and let $k \in e^{-1}(\overline{k})$. Then $e(kan) = \overline{k}\overline{an}$, so $e(g(kan)^{-1}) = 1$.

Thus $g(kan)^{-1} = z \in Z(G)$. By theorem 3.5.14(v) $z \in K$. Thus $g = (zk)an$ so $g \in K \times A \times N$.

To show the map is injective it suffices to show that $K \cap AN = \{1\}$.

If $x \in K \cap AN$ then $e(x) \in \overline{K} \cap \overline{AN} = \{1\}$, so $e(x) = 1$. Write $x = an \in AN$. Then $1 = e(x) = e(an) = e(a)e(n)$ so $e(a) = e(n) = 1$ by the adjoint case. Therefore, as e is injective on A and N, $a = n = 1$ and thus $x = 1$. □

Proposition 3.6.7 *If $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{m} = C_{\mathfrak{k}}(\mathfrak{a})$ then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$*

Proof: It suffices to show that $\mathfrak{h}^{\mathbb{C}}$ is maximal abelian in $\mathfrak{g}^{\mathbb{C}}$. We know $\mathfrak{h}^{\mathbb{C}}$ is abelian. If $Z = X + iY$ commutes with $\mathfrak{h}^{\mathbb{C}}$ then so do X and Y, thus without loss of generality we may consider only X. X commutes with $\mathfrak{a}$ and hence lies in $\mathfrak{a} \oplus \mathfrak{m}$, therefore $\theta(X) \in \mathfrak{a} \oplus \mathfrak{m}$. $X + \theta(X) \in \mathfrak{k}$ so it lies in $\mathfrak{m}$ and commutes with $\mathfrak{t}$ so lies in $\mathfrak{t}$, while $X - \theta(X) \in \mathfrak{a}$ so $X \in \mathfrak{a} \oplus \mathfrak{t}$, so $\mathfrak{h}^{\mathbb{C}}$ is maximal abelian. □

Definition 3.6.8 Borel subgroups

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. A connected Lie subgroup B of G is called a Borel subgroup if there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and an ordering of the corresponding set of roots Φ such that

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{a \in \Phi^+} \mathfrak{g}_a$$

is the Lie algebra of B, where $\mathfrak{g}_a = \{X \in \mathfrak{g} \,|\, \mathrm{ad}(H)X = a(H)X \text{ for all } H \in \mathfrak{h}\}$ and Φ^+ is the set of positive roots of Φ.

Uniqueness results in chapter 2 tell us that different choices of Cartan subalgebras and orderings lead to positive root systems which are conjugates of each other. Thus we deduce that all Borel subgroups of G are conjugate.

Proposition 3.6.9 *Let B be a Borel subgroup of a semisimple Lie group G.*

Then $G = KB = \{kb \,|\, k \in K, b \in B\}$.

Proof: It is clear that $\mathfrak{b} \supseteq \mathfrak{a} \oplus \mathfrak{n}$ so the result follows from the Iwasawa decomposition for Lie groups.

Trivially this tells us that the partition of G into double cosets $K \backslash G / B$ has precisely one element, $K 1_G B = G$.

It is clear that the set of all upper triangular matrices in $\mathfrak{sl}_n(\mathbb{C})$ is a Borel subalgebra of $\mathfrak{sl}_n(\mathbb{C})$. The matrix exponential function maps upper triangular matrices in $\mathfrak{sl}_n(\mathbb{C})$ to upper triangular matrices, as the set of upper triangular matrices is closed under addition and multiplication. It is therefore apparent that the set of upper triangular matrices of $SL_n(\mathbb{C})$ contains a Borel subgroup of $SL_n(\mathbb{C})$. Moreover, the previous proposition tells us that this subgroup should contain a maximal torus, namely the set of all diagonal matrices. Using this, we can deduce that the set of all upper triangular matrices is indeed a Borel subgroup of G. Similarly, we could choose the set of all lower triangular matrices as another Borel subgroup of G. These two subgroups are conjugate via the matrix

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \iddots & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Chapter 4

(Twin) Buildings and (Twin) BN-pairs

Historically groups are understood by considering the ways in which they can act on objects. This section introduces buildings and shows how semisimple Lie groups act on these objects. Following this we continue into the theory of twin buildings and consider necessary properties for a group to act on these objects in an informative way. The main references for this chapter are [Bo02] and [AB08].

4.1 Buildings as W-metric spaces

Definition 4.1.1 Coxeter groups

A symmetric matrix $M = (m_{ij}) \in M_n(\mathbb{Z})$ is called a Coxeter matrix if and only if $m_{ii} = 1$ for all i and $m_{ij} \in \{2, 3, \dots\} \cup \{\infty\}$ whenever $i \neq j$. The Coxeter group W (of type M) is the group with presentation

$$W = \langle s_1, \dots, s_n \,|\, (s_i s_j)^{m_{ij}} \text{ whenever } m_{ij} \neq \infty \rangle$$

Let $S = \{s_1, \dots s_n\}$, then the pair (W, S) is called a Coxeter system of rank $|S|$. Note that this presentation implies that $1_W \notin S$.

If (W, S) is a Coxeter system then we define the Coxeter graph of (W, S) to be the graph with vertex set S and edges between s_i and s_j if and only if $m_{ij} > 2$ that is, these two elements of S do not commute. We give each edge the label m_{ij}. It is clear that this graph is simple, as no vertex can send an edge to itself.

Example 4.1.2 The algebraic Weyl group

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and let $A = (A_{ij})$ be the corresponding abstract Cartan matrix. We know that the algebraic Weyl group $W(\mathfrak{g}, \mathfrak{h})$ is generated by the set of reflections $S = \{S_a \,|\, a \in \Pi\}$, where Π is some system of simple roots, subject to the relations $(s_i s_j)^{m_{ij}}$, where

$$m_{ij} = \begin{cases} 1, & \text{if } i = j \\ 2, 3, 4 \text{ or } 6 & \text{if } A_{ij}A_{ji} = 0, 1, 2 \text{ or } 3 \text{ respectively.} \end{cases}$$

From this we deduce that $W(\mathfrak{g}, \mathfrak{h})$ is a Coxeter group whose rank is equal to the rank of $\mathfrak{g}$. Moreover, the Coxeter diagram of W coincides with the Dynkin diagram of $\mathfrak{g}$ (ignoring the labels on vertices). Therefore figure 2.1.2 provides a presentation of the Weyl group of each complex semisimple Lie group.

Example 4.1.3 Symmetric and dihedral groups

This example covers the Weyl group of type A_{n-1}, namely the symmetric group on n letters, S_n, which is a Coxeter group of rank $n-1$ with presentation

$$S_n = \left\langle s_i = (i\ i{+}1), i \in \{1, 2, \dots, n-1\} \;\middle|\; \begin{array}{ll} (s_i s_j)^3 = 1 & \text{whenever } |i-j| = 1 \\ (s_i s_j)^2 = 1 & \text{otherwise} \end{array} \right\rangle$$

Note that these generators are sufficient to generate S_n as $(1\ n) = (12)^{(2\ 3)(3\ 4)\dots(n-1\ n)}$.

We provide an example to show that there are Coxeter groups which are not algebraic Weyl groups. The dihedral group, D_{2n}, is a Coxeter group of rank 2 with presentation

$$D_{2n} = \langle a, b \mid a^2 = b^2 = (ab)^n \rangle$$

For $n \geq 4$, D_{2n} is not the Weyl group of any complex semisimple Lie algebra as its Coxeter diagram is not present in figure 2.1.2.
In the case $n = 3$ we may enumerate the vertices of a regular hexagon clockwise from 1 to 6 and choose $a = (2\ 6)(3\ 5)$ and $b = (1\ 2)(3\ 6)(4\ 5)$. These are both involutions and $ab = (1\ 2\ 3\ 4\ 5\ 6)$ has order 6.

Lemma 4.1.4 The Cayley graph of (W, S)

Let (W, S) be a Coxeter system and define the Cayley graph Γ to be the graph with vertex set W and an edge from w_1 to w_2 if and only if $w_1(w_2)^{-1} \in S$. Then (W, d) is a metric space, where $d(w_1, w_2) := l(w_1(w_2)^{-1})$ is the smallest integer $q \geq 0$ such that $w_1(w_2)^{-1}$ is the product of a sequence of q elements of S. The map $l : W \to \mathbb{Z}$ is called the length function of W.

Proof: It will suffice to show that the Cayley graph is a simple (undirected) graph and that in this graph $d(w, w')$ corresponds to the length of a shortest path from w to w' as this is a well known metric space.

S is a collection of involutions, so $S = S^{-1}$. Therefore, if $w_1 w_2^{-1} = s \in S$, then $w_2 w_1^{-1} = s^{-1} = s \in S$.

Clearly there is a path from w to w' in this graph of length $l(w(w')^{-1})$ and any shorter path would define a shorter sequence of elements from S whose product is $w(w')^{-1}$ so we can deduce that this is indeed a shortest path. □

Proposition 4.1.5 Equivalent definitions of Coxeter systems

Let W be a group generated by a set S of involutions. The following are equivalent.

(i) *(W, S) is a Coxeter system.*

(ii) *Whenever $w \in W$, $s \in S$ and $l(sw) \leq l(w)$, then given any reduced decomposition $(s_1, \dots, s_q)$ of w there is some $j \in \{1, \dots, q\}$ such that $ss_1s_2 \dots s_{j-1} = s_1s_2 \dots s_j$.*

Condition (ii) of this proposition is called the exchange condition.

Proof: See [Bo02], pages 7-11. □

We will need some key properties of Coxeter groups, which are collected in the next proposition.

Proposition 4.1.6 *Let (W, S) be a Coxeter system of rank n.*

(i) *Let $w \in W$ and $s \in S$, then $l(sw) = l(w) \pm 1$.*

(ii) *If $|W| < \infty$ then there is a unique $w_0 \in W$ such that $l(w_0) \geq l(w)$ for all $w \in W$. (w_0 is called the longest word of W).*

Proof: Using lemma 4.1.4, we see that $d(sw, w) = l(s) = 1 \geq |d(w, 1) - d(sw, 1)| = |l(w) - l(sw)|$, so it suffices to prove that $l(w) \neq l(sw)$.

Suppose $l(sw) \leq l(w)$ and that $(s_1, \dots, s_q)$ is a reduced decomposition of w. As (W, S) satisfies the exchange condition, there is some j, such that $w = ss_1 \dots s_{j-1}s_{j+1} \dots s_q$, so $sw = s_1 \dots s_{j-1}s_{j+1} \dots s_q$ and $l(sw) < l(w)$ as required.

For part (ii) we define an ordering on (W, S) where $w_1 \leq w_2$ if and only if there is some reduced decomposition $(s_1 \dots s_q)$ for w_2 such that $w_1 = (s_{n_1} \dots s_{n_k})$ with $1 \leq n_1 < \dots < n_k \leq q$. This ordering is called the Bruhat ordering and W is a poset with respect to this ordering. As $|W| < \infty$ this ordering must have some maximal element, w_0 and it is clear that such a w_0 can be chosen to be maximal by length. Then $l(w_0) \geq l(w)$ for all $w \in W$.

Suppose w_0 and w_0' are both maximal elements with respect to the Bruhat ordering and that $l(w_0) = l(w_0')$.
Let $(s_1 \dots s_q)$ and $(s_1' \dots s_q')$ be reduced decompositions of w_0 and w_0' respectively.
By definition $l(s_1 w_0') \leq l(w_0')$, so by the exchange condition we could have chosen $s_1' = s_1$.

If $l(s_2 s_2' s_3' \dots s_q') > l(s_2' s_3' \dots s_q') = l(w_0') - 1$ then we can deduce that $s_2 = s_1' = s_1$, as $l(sw_0') < l(w_0')$ for all $s \in S$. But this contradicts the fact that $(s_1 \dots s_q)$ is a reduced decomposition for w_0.

Thus $l(s_2 s_2' s_3' \dots s_q') \leq l(s_2' s_3' \dots s_q')$ so we can use the exchange condition again and deduce that we could have chosen $s_2' = s_2$. Repeating this procedure we see that $w_0 = w_0'$. $\square$

Definition 4.1.7 Buildings

Let (W, S) be the Coxeter system with associated Coxeter matrix M. A building (of type (W, S)) is a pair (Δ, δ) where Δ is a set and $\delta : \Delta \times \Delta \to W$ is a map where the following hold for any pair $x, y \in \Delta$ with $\delta(x, y) = w$

(i) *$w = 1$ if and only if $x = y$,*

(ii) *if $z \in \Delta$ is such that $\delta(y, z) = s \in S$ then $\delta(x, z) \in \{w, ws\}$,*

(iii) *for each $s \in S$ there is a $z \in \Delta$ such that $\delta(x, z) = ws$ and $\delta(y, z) = s$.*

The elements of Δ are called chambers and δ is called the distance function. A building is called spherical if $|W| < \infty$.

If (Δ, δ) is a building, then the chamber graph associated to this building is the graph with vertex set Δ and edges between all pairs of vertices $x, y \in \Delta$ with the property that $\delta(x, y) \in S$. Paths in the chamber graph are called galleries and a shortest path between two chambers is called a minimal gallery.

Example 4.1.8 $\Sigma(W, S)$

Let (W, S) be a Coxeter system. Set $\Delta = W$ and define $\delta : \Delta \times \Delta \to W$ by $\delta(x, y) = xy^{-1}$.

We claim (Δ, δ) is a building of type (W, S) and that the associated chamber graph is the Cayley graph of (W, S).

Proof: Property (i) clearly holds by the uniqueness of inverses.

(ii) holds because $\delta(x, z) = \delta(x, y)\delta(y, z) = ws \in \{w, ws\}$.

For (iii) simply choose $z = sy \in W$. $\square$

A set $A \subseteq \Delta$ is called an apartment if and only if it is isometrically isomorphic to $\Sigma(W, S)$.

Definition 4.1.9 ***s*-panels**

Let (Δ, δ) be a building of type (W, S). For $x \in \Delta$ and $s \in S$ the set

$$P_s(x) := \{y \in \Delta \mid \delta(x, y) \in \{1, s\}\}$$

is called the s-panel of Δ.

A building is said to be thin if every panel has cardinality 2 and thick if every panel has cardinality at least 3.

A panel P is said to belong to an apartment A if $A \cap P \neq \emptyset$.

Continuing with example 4.1.8 we can see that $\Sigma(W, S)$ is a thin building.

As we will see in section 4.4, thick buildings are the ones which will be most important to the study of semisimple Lie groups.

4.2 Buildings as chamber complexes

Definition 4.2.1 Simplicial complexes

A simplicial complex is a partially ordered set (poset) $(X, \leq)$ *satisfying*

(i) *every pair* $x, y \in X$ *have a greatest lower bound,*

(ii) *for each* $x \in X$ *there is some* $r \in \mathbb{N}$ *such that*

$$X_{\leq x} = \{y \in X \mid y \leq x\} \cong \mathcal{P}\{1, \ldots, r\},$$

here $\mathcal{P}(X)$ *denotes the power set of* X*.*

Elements of simplicial complexes are called simplices.

For each $x \in X$ the rank of x, $\text{rank}(x)$ is the value r given by property (ii).

As $X_{\leq x} \cong \mathcal{P}\{1, \ldots, r\}$ for some r we may visualise the set $X_{\leq x}$ as a complete graph Γ on r vertices, in which each element $y \leq x$ defines a complete subgraph of Γ on at most r vertices. Considering each vertex (copy of K_1) as a 0 dimensional point, each copy of K_2 as a line, K_3 as a triangle, K_4 as a tetrahedron and so on, it is sensible to define the dimension of the point x as the dimension of its realisation as a complete graph, so $\dim(x) := r - 1$.

Example 4.2.2 Three simplicial complexes

(i) The poset consisting of all finite subsets of $\mathbb{N}$ under the inclusion ordering is a simplicial complex as every pair of subsets have a greatest lower bound, namely their intersection. If Y is a finite subset of $\mathbb{N}$ of size k, then $X_{\leq Y} \cong \mathcal{P}(Y) \cong \mathcal{P}\{1, \ldots, k\}$ has rank k and dimension $k - 1$.

(ii) Similarly, one can take an r dimensional vector space V with basis $\mathcal{B} = \{v_1, \ldots, v_r\}$ and create a poset of all subspaces of V with basis $\mathcal{B}' \subseteq \mathcal{B}$ ordered by set inclusion. The greatest lower bound of two subspaces is their intersection. Moreover, if $W := \text{span}(\mathcal{B}')$ then $X_{\leq W} \cong \mathcal{P}(\mathcal{B}')$.

(iii) The projective plane $\mathbb{P}^n$ is the poset of all flags of subspaces $(1 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k \subsetneq \mathbb{R}^n)$.

The ordering on $\mathbb{P}^n$ is $(1 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k \subsetneq \mathbb{R}^n) \leq (1 \subsetneq V_0' \subsetneq V_1' \subsetneq \cdots \subsetneq V_l' \subsetneq \mathbb{R}^n)$ if and only if for each $i \in \{0, \ldots k\}$ there is some $j \in \{0, \ldots, l\}$ with $V_i = V_j'$.

The greatest lower bound of two flags is the flag consisting of all common subspaces. Finally, the set of all flags bounded from above by a given flag $(1 \subsetneq V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k \subsetneq \mathbb{R}^n)$ is isomorphic to the power set of $\{V_0, V_1, \ldots, V_j\}$. Similar examples can be made for a projective plane over any field $\mathbb{F}$.

Definition 4.2.3 Chamber complexes

Let $(X, \leq)$ *be a simplicial complex.* $(X, \leq)$ *is called a chamber complex if*

(i) *maximal simplices (chambers) have constant dimension, (the set of chambers is denoted* $\text{Ch}(X)$*),*

(ii) *the chamber graph* $\Gamma = (\text{Ch}(X), E)$ *(defined by* $xy \in E$ *if and only if* $x \cap y$ *has co-dimension 1) is connected.*

A path in the chamber graph is called a gallery.

The notion of a co-dimension is well defined for chamber complexes as they have a well defined dimension, given by the dimension of each maximal chamber.

A chamber complex is called thin if every panel (simplex of co-dimension 1) is contained in at most 2 chambers and thick if every panel is contained in at least 3 chambers.

Example 4.2.4 Continuing with example 4.2.2

(i) The poset consisting of all finite subsets of $\mathbb{N}$ is not a chamber complex as every simplex is contained in a strictly larger simplex. However, the poset given by the power set of some finite set A is a chamber complex with one chamber.

(ii) A more interesting extension to this idea is the poset of all proper subsets of a finite set A. This will be a chamber complex consisting of $|A|$ chambers of rank $|A|-1$, corresponding to the sets $A \setminus \{a\}$ for each $a \in A$. Moreover, this complex is thin as each panel, which is a set $A \setminus \{a,b\}$ of size $|A|-2$ is only contained in the chambers $A \setminus \{a\}$ and $A \setminus \{b\}$.

(iii) The projective plane is a thick chamber complex. This is because a panel consists of a nested sequence $(V_0 = 1 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n - 2 \subsetneq \mathbb{F}^n = V_{n-1})$. Suppose i is the unique value with $\dim(V_i/V_{i-1}) \neq 1$ then this panel is contained in every chamber of the form

$$(V_0 = 1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_i \subsetneq \cdots \subsetneq \mathbb{F}^n = V_{n-1})$$

and there are certainly more than 2 choices for W. If $|F| = \infty$, then there are infinitely many choices for W, so each panel is contained in infinitely many chambers.

Definition 4.2.5 **Coxeter complex**

Let (W,S) be a Coxeter system, the associated Coxeter complex $\Sigma = \Sigma(W,S)$ is given as follows.

Then $\Sigma = \{w\langle T\rangle \mid w \in W,\ T \subseteq S\}$ with order relation $\leq$ given by

$$w_1\langle T_1\rangle \leq w_2\langle T_2\rangle \text{ if and only if } \mathrm{Stab}_W(w_1\langle T_1\rangle) \subseteq \mathrm{Stab}_W(w_2\langle T_2\rangle).$$

For each $T \subseteq S$ we call $\langle T\rangle$ a special subgroup of W, and the cosets of $W/\langle T\rangle$ are called special cosets. Sometimes we will write wW_T for $w\langle T\rangle$.

The chambers of Σ are given by $\{w\}$ for each $w \in W$, the panels by $w\langle\{s\}\rangle = \{w, ws\}$ for each $w \in W$ and each $s \in S$.
The vertices are given by sets $w\langle S \setminus \{s\}$ for each $w \in W$ and $s \in S$ and the unique empty simplex is W (empty because $\mathrm{Stab}_W(W) = \emptyset$).

Example 4.2.6 $\Sigma(W,S)$

Let (W,S) be a Coxeter system, then $\Sigma(W,S)$ is a thin chamber complex of rank $|S|$.

The above comments prove that $\Sigma(W,S)$ is a chamber complex. Let $w\langle\{s\}\rangle = \{w, ws\}$ be a panel. This is contained in precisely two chambers, namely $\{w\}$ and $\{ws\}$. Therefore the claim holds.

Definition 4.2.7 **Simplicial automorphisms**

Let X and Y be simplicial complexes. An order preserving map $X \to Y$ is called simplicial if it does not increase dimension locally and maps vertices to vertices.

Definition 4.2.8 **Links and residues**

Let $(X, \leq)$ be a simplicial complex and let $x \in X$.

The link of x is given by $\mathrm{Lk}_X(x) = \{y \in X \mid x \cap y = \emptyset, \text{ and } x, y \leq z \text{ for some } z\}$

The residue of x is $X_{\geq x} = \{y \in X \mid x \leq y\}$.

It is apparent from the definition that $\mathrm{Lk}_X(x)$ is a subcomplex of X.

Proposition 4.2.9 $\mathrm{Lk}_X(x)$ *and $X_{\geq x}$ are isomorphic as posets.*

Proof: Define $\psi : X_{\geq x} \to \mathrm{Lk}_X(x)$ by $\psi(y)$ is the maximal (with respect to $\leq$) subsimplex of y which satisfies $\psi(y) \cap x = \emptyset$. □

Theorem 4.2.10 *Let $c, c' \in \mathrm{Ch}(\Sigma(W,S))$ and write*

$$\mathrm{Ch}(a) = \{x \in \mathrm{Ch}(\Sigma) \mid d(c,x) < d(c',x)\} \text{ and } \mathrm{Ch}(-a) = \{x \in \mathrm{Ch}(\Sigma) \mid d(c,x) > d(c',x)\}.$$

Then $\mathrm{Ch}(\Sigma) = \mathrm{Ch}(a) \dot\sqcup \mathrm{Ch}(-a)$.

Here the distance metric $d(x, y)$ is given by the length of a shortest xy-path in the Chamber graph.

Proof: It is clear from the above remark that this union is disjoint, so it suffices to show that there are no $x \in \mathrm{Ch}(\Sigma)$ with $d(c, x) = d(c', x)$ but this is just a consequence of the fact that $l(sw) = l(w) \pm 1$ for all $s \in S$ and all $w \in W$. □

We make two remarks here. First $a = \sum_{\leq \mathrm{ch}(a)}$ is called a half-space or root and $-a = \sum_{\leq \mathrm{ch}(-a)}$ is the root opposite to a, secondly, the set $a \cap -a$ is called a wall.

Definition 4.2.11 Buildings

A building is a simplicial complex Δ which is the union of subcomplexes Σ (called apartments), which satisfy the following three conditions.

(B0) Each apartment is a Coxeter complex,
(B1) if $A, B \in \Delta$, then there is an apartment Σ such that $A, B \in \Sigma$,
(B2) if $A, B \in \Delta$ and Σ, Σ' are apartments with $A, B \in \Sigma \cap \Sigma'$
then there is a simplicial isomorphism $\phi : \Sigma \to \Sigma'$ which fixes A and B pointwise.

There are two other conditions which are equivalent to (B2) (given that (B0) and (B1) already hold) which will be of use. These are

(B2′) if $A \in \Delta$, $C \in \mathrm{Ch}(\Delta)$ and Σ, Σ' are apartments with $A, C \in \Sigma \cap \Sigma'$
then there is a simplicial isomorphism $\phi : \Sigma \to \Sigma'$ which fixes A and C pointwise.
(B2″) if Σ and Σ' are apartments which contain a common chamber C,
then there is a simplicial isomorphism $\phi : \Sigma \to \Sigma'$ which fixes $\Sigma \cap \Sigma'$ pointwise.

All apartments are isomorphic by (B2), so we may reasonably talk about the dimension of a building as being the dimension of any of its chambers. A consequence of this property is that all apartments have the same Coxeter matrix. However, the set of apartments is not part of the structure of a building. We define a building to be spherical if and only if the Coxeter group used to define its apartments is finite.

Example 4.2.12 Two buildings

(i) $\Sigma(W, S)$ is a thin building of dimension $|S| - 1$. Property (B0) holds by example 4.2.6 and properties (B1) and (B2) hold vacuously, as $\Sigma(W, S)$ has only one apartment.

(ii) The projective plane $\mathbb{P}^n$ is a thick building of dimension $n - 1$.

To avoid confusion, from this point onwards, buildings in the sense of definition 4.2.11 will be denoted by Δ, and those in the sense of definition 4.1.7 by $(\mathrm{Ch}(\Delta), \delta)$.

Theorem 4.2.13 *The chamber complex and W-metric definitions of a building are equivalent and all the defined notions for each type of building agree.*

Proof: Let Δ be a building in the sense of definition 4.2.11. Each apartment in this building is a copy of $\Sigma(W, S)$ for some Coxeter system (W, S). The set of all chambers of $\Sigma(W, S)$ forms a building in the sense of definition 4.1.7 as was seen in example 4.1.8. Define a metric $\delta : \mathrm{Ch}(D) \times \mathrm{Ch}(D) \to W$ by $\delta(C, D) = w$ where w is the distance between C and D in any apartment containing both these chambers. This is well defined as any two apartments containing C and D are isomorphic by condition (B2). The remaining properties follow from the fact that $(\mathrm{Ch}(\Sigma(W, S)), \delta)$ is a building in the sense of definition 4.1.7.

The reverse implication is more difficult, so the reader is referred to [AB08], section 5.6. □

Using the first implication we can define a metric d on a building Δ where

$$d(x, y) = \min\{l(\delta(C, D)) \mid x \leq C,\, y \leq D\}.$$

4.3 Solomon-Tits theorem

Later results will require a building to be simply connected. With this in mind we will now state the Solomon-Tits theorem. A complete proof would need knowledge of basic homotopy theory. Since this is not assumed, these details will be stated, but their proofs will be omitted. Using theorem 4.2.13 we will interchange between the two definitions.

Definition 4.3.1 **Bouquet of spheres**

Let $(X_i)_{i\in I}$ be a collection of topological spaces and for each $i \in I$, let $x_i \in X_i$. The wedge sum of the X_i is

$$\bigvee_{i\in I} X_i := \bigsqcup_{i\in I} X_i \Big/ \sim,$$

where $\sim$ is the equivalence relation, $a \sim b$ if and only if $a = x_i$ and $b = x_j$ for some $i, j \in I$. The point x_i is called the base point of X_i. These base points are equivalent in $\bigvee_{i\in I} X_i$ so we denote the unique base point of this set by $\overline{x}$.

If each X_i is homeomorphic to the sphere $S^n = \{x \in \mathbb{R}^{n+1} \,|\, |x| = 1\}$, then $\bigvee_{i\in I} X_i$ is called a bouquet of n-spheres.

Proposition 4.3.2 *A bouquet of n-spheres is simply connected if and only if $n \geq 2$*

Proof: S^n is simply connected if and only if $n \geq 2$. Let X be a bouquet of spheres. It is clear that X is path connected. Consider a continuous curve $f : S^1 \to X$. If the image $f(S^1)$ avoids the base point of X then it is entirely contained in some copy of S^n so is in the same homotopy class as the identity curve $f(x) = \overline{x}$. Here we are using the fact that S^n is simply connected, so its fundamental group is trivial.

Suppose now that $\overline{x} \in f(S^1)$. If $f(x) = \overline{x}$ for all x then f is the identity curve, so suppose there is some $a \in S^1$ with $f(a) \neq \overline{x}$. As f is continuous there is a connected open subset U of S^1, such that $\overline{x} \notin f(U)$, but $f(x) = \overline{x}$ for all $x \in \overline{U} \setminus U$. But then $f(\overline{U})$ is contained in some copy of S^n so is in the same homotopy class as the curve f' which is the identity map on $\overline{U}$ and equal to f elsewhere.

As this process can be done for any point not mapped to $\overline{x}$, we deduce that f lies in the same homotopy class as the identity curve. Thus X has trivial fundamental subgroup and is therefore simply connected. □

To ease the notation in the following results we denote the set $\Delta_{\leq A} = \{X \in \Delta \,|\, X \leq A\}$ by $\overline{A}$.

Lemma 4.3.3 *Let Δ be a building. Fix some chamber $C \in \mathrm{Ch}(\Delta)$ and some integer $d \geq 1$. Let $\mathcal{D}$ be a set of chambers with the following two properties.*

(i) *$d(C, D) \leq d$ for all $D \in \mathcal{D}$,*

(ii) *if $d(C, D) < d$ then $D \in \mathcal{D}$.*

Let Δ' be the subcomplex of Δ generated by $\mathcal{D}$ and let D be a chamber of Δ with $D \notin \mathcal{D}$ and $d(C, D) = d$. Then

$$\overline{D} \cap \Delta' = \bigcup_{A\in\mathcal{P}} \overline{A},$$

where $\mathcal{P}$ is the set of all panels A of D with $d(C, A) < d$ (i.e. A lies on some chamber strictly closer to C than D is).

Moreover, the set $\mathcal{P}$ contains all the panels of D if and only if Δ is spherical and of diameter d.

Proof: Firstly, we claim that $\overline{D} \cap \Delta' = \{B < D \,|\, d(C, B) < d\}$.

The inclusion $\overline{D} \cap \Delta' \supseteq \{B < D \,|\, d(C, B) < d\}$ is obvious, by assumption (ii). Suppose $B \in \overline{D} \cap \Delta'$. Then there is some chamber $D' \in \mathcal{D}$ with $B < D'$ and $d(C, B) \leq d(C, D') \leq d = d(C, D)$, by assumption (i). If $d(C, B) = d$, then $d(C, B) = d(C, D') = d(C, D)$, so $D = D'$, which contradicts the fact that $D \notin \mathcal{D}$ and $D' \in \mathcal{D}$.

Secondly, we claim that if $B < D$ and $d(C, B) < d$ then there is some panel A of D with the property that $B \leq A$ and $d(C, A) < d$.

This claim can be verified by considering a minimal gallery from C to D and choosing the panel which connects D to the chamber preceding it in this minimal gallery.

Combining the two claims obtains the main result.

If $\mathcal{P}$ contains all the panels of D and $\delta(C, D) = w$ then $l(sw) < l(w)$ for all $s \in S$. This can only happen if W is finite and w is the longest word in W. If Δ is spherical and of diameter d, then each panel A of D lies in some chamber D_A with $d(C, D_A) < d(C, D) = D$, so $A \in \mathcal{P}$. □

Theorem 4.3.4 Solomon-Tits theorem

Let Δ be a building of rank n. Then Δ is a bouquet of $(n-1)$ spheres.

By proposition 4.3.2, the Solomon-Tits theorem says that Δ is simply connected if and only if it has rank at least 3.

Proof: Pick some chamber $C \in \mathrm{Ch}(\Delta)$, choose $d = l(w_0)$, where w_0 is the unique longest word of (W, S) and set $\mathcal{D} = \{X \in \mathrm{Ch}(\Delta) \,|\, d(X, C) < d\}$. Let Δ' be the subcomplex of Δ generated by $\mathcal{D}$.

By lemma 4.3.3, we know that $\overline{D} \cap \Delta' = \bigcup_{A \in \mathcal{P}} \overline{A}$, where $\mathcal{P}$ is the set of all the panels of D. Using this it can be proved that any continuous curve on Δ' is homotopy equivalent to a continuous curve on the single chamber C. Thus Δ has the same fundamental subgroup as the simplicial complex obtained by contracting all chambers of Δ' to a single chamber. Reintroducing the chambers at distance d from C, we notice that each of these chambers shares a panel with exactly one chamber of Δ which is closer to C than it. This chamber must lie in Δ' by construction, so this opposite chamber has the fundamental subgroup of a sphere with base point given by a chosen point in the contraction of Δ'. We chose a generic opposite chamber so it follows that the homotopy type of Δ is that of a bouquet of $(n-1)$-spheres (as $n-1$ is the dimension of each chamber). □

4.4 BN-pairs

Definition 4.4.1 BN**-pairs**

Let G be a group and let B, N be subgroups of G with $B \cap N \trianglelefteq N$. Define $T := B \cap N$, $W := N/T$ and let S be a generating set for W. The tuple (G, B, N, S) is called a BN-pair if

(i) *G is generated by $B \cup N$,*

(ii) *for each $w \in W$ and $s \in S$, $wBs \subseteq BwB \cup BwsB$,*

(iii) *for each $s \in S$, $sBs^{-1} \not\subseteq B$.*

If (G, B, N, S) is a BN-pair then we say that G admits a BN-pair.

It is natural to ask what additional structure can be guaranteed on a group which admits a BN-pair. The following proposition begins to answer this question.

Proposition 4.4.2 *Let (G, B, N, S) be a BN-pair. Then*

(i) *there is a bijection between W and the double cosets $B \backslash G / B$. In particular G has the Bruhat decomposition*
$$G = \bigsqcup_{w \in W} BwB,$$

(ii) *(W, S) is a Coxeter system whenever $|S| < \infty$, in particular, $s = s^{-1}$ for all $s \in S$,*

(iii) $BwBsB = \begin{cases} BwsB, & \text{if } l(ws) > l(w) \\ BwB \cup BwsB, & \text{if } l(ws) < l(w) \end{cases}$

Note that $BwB = B(Tn)B = (BT)nB = BnB$ is a well defined double coset.

Proof: See [Br88] pages 107-110. □

Definition 4.4.3 Group action on a building

A group G is said to act on a building $(\mathrm{Ch}(\Delta), \delta)$ *of type* (W, S) *if there is a δ-preserving homomorphism $\phi : G \to \mathrm{Sym}(\Delta)$. In other words, ϕ is a homomorphism and for all $g \in G$ and all $x, y \in \Delta$* $\delta((g\phi)(x), (g\phi)(y)) = \delta(x, y)$.

G is said to act transitively on $(\mathrm{Ch}(\Delta), \delta)$ *if for any pair $x, y \in \Delta$ there is some $g \in G$ such that* $(g\phi)(x) = y$.

Finally, G acts strongly transitively on $(\mathrm{Ch}(\Delta), \delta)$ *if for each $w \in W$, the action of G is transitive on the set of pairs*

$$(\Delta \times \Delta)_w := \{(x, y) \in \Delta \times \Delta \mid \delta(x, y) = w\}.$$

The next theorem will be of great importance to us, as it will show that groups which admit a BN-pair act strongly transitively on an associated thick building. Moreover, the proof is constructive so enables us to study the group via the building. This will be a necessary tool in later chapters.

Theorem 4.4.4 *Let G be a group which admits a BN-pair. Then there is an associated thick building on which G acts strongly transitively.*

Proof: We define the building $(\mathrm{Ch}(\Delta), \delta)$ using the Bruhat decomposition guaranteed by proposition 4.4.2. Let $\Delta := B\backslash G$ be the set of left cosets of G with respect to B and define $\delta : \Delta \times \Delta \to W$ by

$$\delta(gB, hB) = w \text{ if and only if } Bg^{-1}hB = BwB.$$

The Bruhat decomposition tells us that such an operation is well defined. We must check that δ satisfies the three conditions necessary in definition 4.1.7 and show that G acts strongly transitively on $(\mathrm{Ch}(\Delta), \delta)$.

(i) Suppose $\delta(gB, hB) = 1$, then $Bg^{-1}hB = B$, so $g^{-1}h \in B$ and $h \in gB$ which implies that $gB = hB$.

(ii) Now assume $\delta(gB, hB) = w$ and suppose $kB \in G/B$ is such that $\delta(hB, kB) = s \in S$, then $Bg^{-1}hB = BwB$ and $Bh^{-1}kB = BsB$. Therefore, $Bg^{-1}kB \subseteq (Bg^{-1}hB)(Bh^{-1}kB) = BwBBsB = BwBsB \subseteq BwB \cup BwsB$.
Thus $Bg^{-1}kB = BwB$ or $BwsB$ as required.

(iii) Suppose for every choice of kB with $\delta(hB, kB) = s \in S$, $Bg^{-1}kB = BwB$. Then $BwBsB \subseteq BwB$ and therefore $ws \in BwB$ contradicting the Bruhat decomposition. Therefore there is some coset kB with $\delta(hB, kB) = s \in S$ and $Bg^{-1}kB = BwsB$, so $\delta(gB, kB) = ws$ as required.

It remains to prove that G acts strongly transitively on this building.
Obviously, G acts transitively on $B\backslash G$. It therefore suffices to show that there exists some $b \in B$ such that $bgB = hB$ whenever $\delta(B, gB) = \delta(B, hB) = w$. By definition $BgB = BhB = BwB$, so there is clearly some $b \in B$ with the required property. □

Due to the equivalence of the two definitions of buildings 4.2.13 and the fact that the two definitions of strongly transitive group actions agree, we deduce from this theorem that G acts strongly transitively on the equivalent building Δ.

Equivalently, we could have defined the building as the set of all right cosets $\{Bg \mid g \in G\}$ and defined

$$\delta(Bg, Bh) = w \text{ if and only if } Bgh^{-1}B = BwB.$$

Theorem 4.4.5 *Let G be a group which acts strongly transitively on some thick building Δ. Then G admits a BN-pair.*

Proof: We consider G as a strongly transitive automorphism group on Δ and fix some chamber C in some apartment Σ. Define the three subgroups

$$\begin{aligned} B &= \{g \in G \mid gC = C\} \\ N &= \{g \in G \mid g\Sigma = \Sigma\} \\ T &= \{g \in G \mid g \text{ fixes } \Sigma \text{ pointwise}\} \end{aligned}$$

It is clear that $T \trianglelefteq N$ as it is the kernel of the homomorphism $\alpha : N \to W$ induced by the action of N on the apartment $\Sigma = \Sigma(W, S)$. Moreover, (G, B, N, S) is a BN-pair for G. The full details of the proof can be found in [Br88], pages 102-106. □

4.5 Twin buildings and twin BN-pairs

The previous section shows how we may associate a building to any group which admits a BN-pair, in particular to semisimple Lie groups. This will be useful in the proof of Borovoi's theorem in section 5.3. In section 8.1 we will extend this result to Kac-Moody groups which admit a twin BN-pair. For this we will need to generalise the results from the previous section to determine a natural structure on which these groups act. It will be shown that twin buildings are the natural objects to consider.

Definition 4.5.1 Twin building

A triple $((\mathrm{Ch}(\Delta_+), \delta_+)$ *and* $(\mathrm{Ch}(\Delta_-), \delta_-), \delta^*)$ *consisting of two buildings* $(\mathrm{Ch}(\Delta_+), \delta_+)$ *and* $(\mathrm{Ch}(\Delta_-), \delta_-)$ *of type* (W, S) *and a codistance function* $\delta^* : \mathrm{Ch}(\Delta_+) \times \mathrm{Ch}(\Delta_-) \cup \mathrm{Ch}(\Delta_-) \times \mathrm{Ch}(\Delta_+) \to W$ *is called a twin building of type* (W, S) *if for each* $\varepsilon \in \{+, -\}$ *and each pair* $(x, y) \in \mathrm{Ch}(\Delta_\varepsilon) \times \mathrm{Ch}(\Delta_{-\varepsilon})$ *with* $\delta^*(x, y) = w$,

(i) $\delta^*(x, y) = \delta^*(y, x)^{-1}$,

(ii) *if* $x' \in \Delta_\varepsilon$ *is such that* $\delta_\varepsilon(x', x) = s \in S$ *and* $l(ws) < l(w)$, *then* $\delta^*(x', y) = ws$,

(iii) *for any* $s \in S$ *there is some* $x' \in \Delta_\varepsilon$ *such that* $\delta_\varepsilon(x', x) = s$ *and* $\delta^*(x', y) = ws$.

Two chambers $x \in \mathrm{Ch}(\Delta_+)$ and $y \in \mathrm{Ch}(\Delta_-)$ are said to be opposite if $\delta^*(x, y) = 1$.

Example 4.5.2 A spherical twin building of type (W, S)

Let $(\mathrm{Ch}(\Delta), \delta)$ be a spherical building of type (W, S) and let w_0 be the unique longest word in W with respect to S. Define $(\mathrm{Ch}(\Delta_+), \delta_+), (\mathrm{Ch}(\Delta_-), \delta_-)$ to be two disjoint copies of $(\mathrm{Ch}(\Delta), \delta)$ and define the codistance δ^* by

$\delta^* : \mathrm{Ch}(\Delta_+) \times \mathrm{Ch}(\Delta_-) \to W$ is given by $\delta^*(x, y) = \delta_+(x, y)w_0$
$\delta^* : \mathrm{Ch}(\Delta_-) \times \mathrm{Ch}(\Delta_+) \to W$ is given by $\delta^*(x, y) = w_0\delta_-(x, y)$.

Then $((\mathrm{Ch}(\Delta_+), \delta_+), (\mathrm{Ch}(\Delta_-), \delta_-), \delta^*)$ is a twin building of type (W, S).

In light of this we may view twin buildings as a generalisation of spherical buildings.

Definition 4.5.3 Group action on a twin building

A group G *is said to act on a twin building* $\Delta = ((\mathrm{Ch}(\Delta_+), \delta_+), (\mathrm{Ch}(\Delta_-), \delta_-), \delta^*)$ *if it acts simultaneously on the two sets* $\mathrm{Ch}(\Delta_+)$ *and* $\mathrm{Ch}(\Delta_-)$ *and also preserves* δ_+, δ_- *and* δ^*.

In particular this means that for all $x, x' \in (\mathrm{Ch}(\Delta_+)$, $y, y' \in (\mathrm{Ch}(\Delta_-)$ *and* $g \in G$,

(i) $\delta_+((g\phi)x, (g\phi)x') = \delta_+(x, x')$,

(ii) $\delta_-((g\phi)y, (g\phi)y') = \delta_-(y, y')$,

(iii) $\delta^*((g\phi)x, (g\phi)y) = \delta^*(x, y)$,

where ϕ *is the associated group action.*

To make sure the twin building is a suitable structure for the group to act on we require stronger properties on such an action, namely, that it is strongly transitive. There are many equivalent definitions of this, two of which will be collected in the next proposition.

Proposition 4.5.4 Equivalent definitions of strongly transitive

Let G *be a group which acts on a twin building* $\Delta = ((\mathrm{Ch}(\Delta_+), \delta_+), (\mathrm{Ch}(\Delta_-), \delta_-), \delta^*)$. *The following are equivalent.*

For each $w \in W$, G *acts transitively on the set*

$$\{(x, y) \in \mathrm{Ch}(\Delta_+) \times \mathrm{Ch}(\Delta_-) \mid \delta^*(x, y) = w\}.$$

For $\varepsilon \in \{+,-\}$, G acts transitively on Δ_ε and for each $w \in W$ and $x \in \Delta_\varepsilon$, B^ε acts transitively on the set

$$\{y \in \Delta_{-\varepsilon} \mid \delta^*(x,y) = w\}.$$

Proof: See [AB08], lemma 6.70. □

We say a group G acts strongly transitively on a twin building if and only if the action satisfies either of the properties given in proposition 4.5.4.

Definition 4.5.5 Twin BN-pairs

Let G be a group and let B^+, B^- and N be subgroups of G such that $B^+ \cap N = B^- \cap N = T \trianglelefteq N$ and there is some subset $S \subseteq W := N/T$ such that (G, B^ε, N, S) is a BN-pairs of G for $\varepsilon \in \{+,-\}$. Then the system (G, B^+, B^-, N, S) is called a twin BN-pair if

(i) *for each $w \in W$ and $s \in S$ with $l(ws) < l(w)$, $B^\varepsilon w B^{-\varepsilon} s B^{-\varepsilon} = B^\varepsilon ws B^{-\varepsilon}$,*

(ii) *for each $s \in S$, $sB^- \cap B^+ = \emptyset$.*

Proposition 4.5.6 Properties of groups which admit twin BN-pairs

Let (G, B^+, B^-, N, S) be a twin BN-pair, then

(i) *for each $\varepsilon \in \{+,-\}$, each $s \in S$ and $w \in W$,*

$$B^\varepsilon w B^{-\varepsilon} s B^{-\varepsilon} \subseteq B^\varepsilon ws B^{-\varepsilon} \cup B^\varepsilon w B^{-\varepsilon},$$

(ii) *G has a Birkhoff decomposition, so*

$$G = \bigsqcup_{w \in W} B^\varepsilon w B^{-\varepsilon}, \qquad \text{for each } \varepsilon \in \{+,-\},$$

Proof:

(i) We consider the case $\varepsilon = +$ and $l(w) < l(ws)$, the other three cases can be handled similarly. Recall that as (G, B^-, N, S) is a BN-pair, $B^- s B^- s B^- = B^- \cup B^- s B^-$. Then

$$\begin{aligned} B^+ w B^- s B^- &= (B^+ w B^-) s B^+ \\ &= (B^+ ws B^- s B^-) s B^- && \text{by definition 4.5.5(i)} \\ &= B^+ ws (B^- s B^- s B^-) \\ &= B^+ ws (B^- \cup B^- s B^-) && \text{as } (G, B^-, N, S) \text{ is a } BN\text{-pair} \\ &= B^+ ws B^- \cup B^+ ws B^- s B^- \\ &= B^+ ws B^- \cup B^+ w B^- && \text{by definition 4.5.5(i).} \end{aligned}$$

(ii) We will prove the case $\varepsilon = +$. First we claim that $B^+ w B^- \neq B^+ B^-$ whenever $1_W \neq w \in W$. Assume for a contradiction that this is false for some w. As $w \neq 1$ there is some s such that $l(ws) < l(w)$. Then by part (i),

$$B^+ ws B^- = B^+ B^- s B^- = B^+ B^- \cup B^+ s B^-.$$

The left hand side is one double coset and the right hand side is the union of two distinct double cosets by definition 4.5.5(ii), which is a contradiction.

Next we show that $B^+ v B^- = B^+ w B^-$ implies that $v = w$ by induction on $\min\{l(v), l(w)\}$ which we assume to be $l(v)$.
The case $l(v) = 0$ is covered by the previous claim, so assume $l(v) > 0$ and choose $s \in S$ such that $l(vs) < l(v)$. If $B^+ v B^- = B^+ w B^-$, then using the same trick as earlier, we deduce that $B^+ vs B^- = B^+ w B^- s B^-$. There is only one double coset on the left hand side, so it follows that the right hand side is simply the double coset $B^+ ws B^-$.
By the induction hypothesis, $vs = ws$, so $v = w$.

Finally, we show $G = \bigcup_{w \in W} B^+ w B^-$. By part (i) the union is closed under left multiplication by both B^+ and N, so is closed under left multiplication by $B^+ N B^+$ which is the whole of G by the Bruhat decomposition. □

Theorem 4.5.7 *Let G be a group which admits a twin BN-pair (G, B^+, B^-, N, S). Then G acts strongly transitively on an associated twin building $\Delta = ((\mathrm{Ch}(\Delta_+), \delta_+), (\mathrm{Ch}(\Delta_-), \delta_-), \delta^*)$.*

We know that the two BN-pairs act strongly transitively on associated thick buildings $(\mathrm{Ch}(\Delta_+), \delta_+)$ and $((\mathrm{Ch}(\Delta_-), \delta_-)$ respectively. It remains to show that there is a suitable codistance function δ^*. For $\varepsilon \in \{+, -\}$, set

$$\delta^*(gB^\varepsilon, hB^{-\varepsilon}) = w, \text{ if and only if }, B^\varepsilon gh^{-1}B^{-\varepsilon} = B^\varepsilon wB^{-\varepsilon}.$$

This is well defined as G has a Birkhoff decomposition. There are three properties on δ^* to check.

(i) It is clear from the definition that $\delta^*(x, y) = \delta^*(y, x)^{-1}$.

(ii) Now assume $\delta^*(gB, hB) = w$ and suppose $kB \in G/B$ is such that $\delta_-(gB, kB) = s \in S$, with $l(ws) < l(w)$. Then $B^+g^{-1}hB^- = B^+wB^-$ and $B^-g^{-1}kB^- = B^-sB^-$.
Therefore,

$$\begin{aligned} B^+g^{-1}kB^- &\subseteq (B^+g^{-1}hB^-)(B^-h^{-1}kB^-) \\ &= B^+wB^-B^-sB^- = B^+wB^-sB^- \\ &= B^+wsB^-, \text{ as } l(ws) < l(w). \end{aligned}$$

Both the left hand side and right hand side are double cosets in the Birkhoff decomposition, so $B^+g^{-1}kB^- = B^+wsB^-$ as required.

(iii) Suppose for every choice of kB^- with $\delta_-(hB^-, kB^-) = s \in S$, $B^+g^{-1}kB^- = B^+wB^-$. Then $B^+wB^-sB^- \subseteq B^+wB^-$ and therefore $ws \in B^+wB^-$ contradicting the Birkhoff decomposition. Therefore there is some coset kB^- with $\delta_-(hB^-, kB^-) = s \in S$ and $B^+g^{-1}kB^- = B^+wsB^-$, so $\delta^*(gB^+, kB^-) = ws$, as required. □

Let G be a group which admits an abstract root system (in the sense of definition 2.3.4 without the condition that Φ spans $(\mathfrak{h}_0)^*$.) A pair of roots $\{a, b\} \subseteq \Phi$ is said to be prenilpotent if there are elements $w_1, w_2 \in W$ such that

$$\{w_1(a), w_1(b)\} \subseteq \Phi^+ \quad \text{and} \quad \{w_2(a), w_2(b)\} \subseteq \Phi^-$$

or equivalently that the intersections of half spaces $a \cap b$ and $(-a) \cap (-b)$ are both nonempty. For a prenilpotent pair $\{a, b\}$ we define

$$[a, b] := \{c \in \Phi \mid a \cap b \subseteq c \text{ and } (-a) \cap (-b) \subseteq (-c)\}$$

and set $(a, b) := [a, b] \setminus \{a, b\}$.

Definition 4.5.8 **Twin root datum (Root group datum)**

Let G be a group with abstract root system $\Phi = \Phi^+ \sqcup \Phi^-$. A system $(G, (U_a)_{a\in\Phi})$ is called a twin root datum if it satisfies the following axioms. (For ease of notation we define $U^\varepsilon = \langle U_a \mid a \in \Phi^\varepsilon \rangle$).

(i) *Each U_a is a non-trivial subgroup of G.*

(ii) *$G = \langle U_a \mid a \in \Phi \rangle$.*

(iii) *If $b \in \Pi$ (the set of simple roots), then $U_b \not\subseteq U^-$.*

(iv) *If $a \in \Phi$ and $u \in U_a \setminus \{1\}$, then there exist elements $u_1, u_2 \in U_{-a}$ such that u_1uu_2 conjugates U_b to $U_{s_a(b)}$ for each $b \in \Phi$.*

(v) *If the pair $\{a, b\} \subseteq \Phi$ is prenilpotent then the commutator group $[U_a, U_b] \subseteq \langle U_c \mid c \in (a, b) \rangle$.*

(vi) *If $b \in \Pi$ then there is some b' in the root subsystem generated by b such that $U_a \subseteq U_{b'}$ for all a in the root system generated by b.*

It can be shown that the elements chosen in part (iv) of this definition are uniquely determined so the map $\mu(u) = u_1uu_2$ is well defined.

The groups U_a are called root subgroups of G, explaining the description of such an object as a root group datum. The following proposition justifies our choice of describing this object as a twin root datum.

Proposition 4.5.9 Groups with a twin root datum admit a twin BN-pair

If G is a group which admits a twin root datum $(G, (U_a)_{a\in\Phi})$ then (G, B^+, B^-, N, S) is a twin BN-pair, where

$$T := \bigcap_{a\in\Phi} N_G(U_a)$$

$$N := \langle T \cup \{\mu(u) \,|\, u \in U_a \setminus \{1\},\, a \in \Phi\}\rangle \text{ and } B^\varepsilon = TU^\varepsilon.$$

We follow the proof given in [Ma07].

Proof: $U^+ \subseteq B^+$ and $T \subseteq N$ so G is certainly generated by $B^+ \cup N$.

Next we wish to prove that $B^+ \cap N = T$ and T is normal in N. T is the intersection of the normalisers of the U_a so it follows that $tU_at^{-1} = U_a$ for all $a \in \Phi$ and all $t \in T$. Definition 4.5.8(iv) tells us that the action of conjugating by elements of N permutes the root subgroups. However, $B^+ \cap N$ is the subgroup of N consisting of all those elements generated by T and the positive root subgroups, so must fix the positive root-spaces under the conjugation action. Therefore $B^+ \cap N$ is normal in N and $B^+ \cap N = T$.

The group $W := N/T$ acts on the set of root groups by conjugation. Again by property (iv), for each $a \in \Phi$ there is some element $w \in W$ such that for some suitable b,

$$w^2U_b = wU_{s_a(b)} = U_{s_a(s_a(b))} = U_b.$$

Thus W is generated by involutions.

The remaining two properties are easy consequences of the definition of an abstract root system. The same arguments hold with B^+ replaced by B^- so we have shown that G admits two BN-pairs. The remaining axioms for a twin BN-pair are more technical and are therefore omitted. They may be found in [Ti87]. □

4.6 Semisimple Lie groups with BN-pairs and twin BN-pairs

In this section we will show that certain semisimple Lie groups admit BN-pairs and twin BN-pairs and therefore act strongly transitively on associated buildings and twin buildings respectively. We will follow the method given in [AB08], section 7.9.2, specifying the construction to Lie groups.

Let G be a connected semisimple Lie group, whose Lie algebra $\mathfrak{g}$ is a complex semisimple Lie algebra. Such semisimple Lie groups will be referred to as complex semisimple Lie groups. Such groups admit an abstract root system $\Phi(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

To each root $a \in \Phi$ there is a corresponding copy of $\mathfrak{sl}_2(\mathbb{C})$ contained in $\mathfrak{g}$, which is generated by the set $\{E_a, E_{-a}, H_a\}$, where $E_a \in \mathfrak{g}_a$, $E_{-a} \in \mathfrak{g}_{-a}$ and $H_a \in \mathfrak{h}$.

The exponential mapping takes this set to a copy of $SL_2(\mathbb{C})$ in G which will be denoted by G_a.

We denote by U_a the image of the set of all upper unitriangular matrices in $SL_2(\mathbb{C})$ contained in G_a.

We claim that $(G, (U_a)_{a\in\Phi})$ is a twin root datum. As with the definition we will define the sets $U^\varepsilon = \langle U_a \,|\, a \in \Phi^\varepsilon\rangle$, for ease of notation.

(i) Each U_a is clearly nontrivial as $U_a \cong (\mathbb{C}, +)$.

(ii) $G_a = \langle U_a, U_{-a}\rangle$, so all copies of G_a are contained in $\langle U_a \,|\, a \in \Phi\rangle$. G is connected, so the exponential map is onto by proposition 3.3.7(iii). Therefore $\langle U_a \,|\, a \in \Phi\rangle = G$.

(iii) U^- is generated by lower triangular matrices, so if $b \in \Pi \subseteq \Phi^+$, then U_b is a group of upper triangular matrices.
$U^- \neq G$, so it follows that $U_b \nsubseteq U^-$.

Finally, we note that as the root system generated by any root b is simply the pair $\{b, -b\}$, property (vi) holds trivially.

For the remaining two properties we introduce some additional notation.

We define $x_a : \mathbb{C} \to U_a$ to be the obvious isomorphism and define the following elements of G,

$$\begin{aligned} m_a(z) &:= x_a(z)x_{-a}(-z^{-1})x_a(z), \\ m_a &:= m_a(1) \quad \text{and} \\ h_a(z) &:= m_a(z)(m_a)^{-1}. \end{aligned}$$

for all $a \in \Phi$ and all $z \in \mathbb{C} \setminus \{0\}$.

From [St68], lemma 28(a), we see that $h_a : \mathbb{C}^\times \to U_a$ is in fact a homomorphism, so $\{h_a(z) \mid z \in \mathbb{C}^\times\}$ is a subgroup of G which is isomorphic to the multiplicative group $\mathbb{C}^\times$.

We now determine conjugation relations on the elements m_a and $h_a(z)$. The first two relations are

$$m_a x_b(z)(m_a)^{-1} = x_{s_a(b)}(\pm z) \qquad \text{and}$$

$$h_a(z)x_b(z')h_a(z) = x_b(z^{\langle b,a^\vee\rangle}z')$$

for all $a, b \in \Phi$, all $z \in \mathbb{C}^\times$ and all $z' \in \mathbb{C}$. (Here, $a^\vee := \frac{2}{\langle a,a\rangle}$).

Proofs of both these results can be found in [St68] on page 30. We will soon see that the exact nature of the $\pm$ in the first equation is soon made irrelevant for our purposes.

Combining these two results with the definition of $h_a(z)$, we obtain the new formula

$$m_a(z)x_a(z')m_a(z)^{-1} = x_{s_a(b)}(z^{\langle s_a(b),a^\vee\rangle}z'). \qquad (\dagger)$$

Setting $a = b$ we see that

$$h_a(z)x_a(z')h_a(z)^{-1} = x_a(z^2 z'),$$

and therefore we obtain the first important commutator formula

$$[h_a(z), x_a(z')] = x_a((z^2-1)z').$$

From this we deduce that $[h_a(z), U_a] = U_a$, provided $z \notin \{0, \pm 1\}$.

Secondly, we deduce conjugation relations on the generators $x_a(z)$. Let a and b be two nonproportional roots and let $z, z' \in \mathbb{C}$, then the commutator relationship has the form

$$[x_a(z), x_b(z')] = \prod_{i,j\in\mathbb{N}} \left(\prod_{ia+jb\in\Phi} x_{ia+jb}(c_{ij}z^i(z')^j) \right). \qquad (\ddagger)$$

Certain details are omitted here, as an ordering must be chosen on the elements in the product (as they do not commute). The integers c_{ij} depend on a, b and the order chosen.

We can now complete the proof that G admits a twin root datum.

Property (v) follows from ($\ddagger$) and the fact that $\{ia + bj \mid i, j \in \mathbb{Z}\} \cap \Phi \subseteq (a, b)$.

Finally, to prove property (iv), let $u = x_a(z) \in U_a \setminus \{1\}$, then choosing $u_1 = m_{-a}(-z^{-1})$ and $u_2 = (u_1)^{-1}$ will suffice, by ($\dagger$). $\square$

This result combined with proposition 4.5.9 is sufficient to show that every complex semisimple Lie group admits a twin BN-pair and therefore admits a BN-pair and acts strongly transitively on both an associated building and an associated twin building. We will use this result in the next chapter.

Chapter 5

Borovoi's theorem

The first application we make of the theory developed in the preceding chapters is to prove a topological version of Borovoi's theorem. The main reference for this chapter is [GGH09], sections 1-3.

5.1 Introductory Category theory

Definition 5.1.1 Categories

A pair $\mathbb{I} = (\text{ob}(\mathbb{I}), \text{mor}(\mathbb{I}))$ *is called a category if the following properties hold.*

(i) *Every* $f \in \text{mor}(\mathbb{I})$ *has a unique source object* $a \in \text{ob}(\mathbb{I})$ *and a unique target object* $b \in \text{ob}(\mathbb{I})$*. This is denoted by* $f : a \to b$*. The collection of all* $f \in \text{mor}(\mathbb{I})$ *with* $f : a \to b$ *is denoted by* $\text{mor}(a, b)$*.*

(ii) *If* $f \in \text{mor}(a, b)$ *and* $g \in \text{mor}(b, c)$ *then there is some* $h \in \text{mor}(a, c)$ *with* $h = g \circ f$*. (*$\text{mor}(\mathbb{I})$ *is closed under composition).*

(iii) *If* $f \in \text{mor}(a, b)$*,* $g \in \text{mor}(b, c)$ *and* $h \in \text{mor}(c, d)$ *then* $h \circ (g \circ f) = (h \circ g) \circ f$*. (Composition in* $\text{mor}(\mathbb{I})$ *is associative).*

(iv) *For every* $x \in \text{ob}(\mathbb{I})$ *there is some* $\iota_x \in \text{mor}(x, x)$ *such that for every* $f \in \text{mor}(a, b)$*,* $\iota_b \circ f = f = f \circ \iota_a$*.*

$\text{ob}(\mathbb{I})$ is called the class of objects of $\mathbb{I}$ and $\text{mor}(\mathbb{I})$ is called the class of morphisms of $\mathbb{I}$. A category is called small if $\text{ob}(\mathbb{I})$ is a set.

Example 5.1.2 Important categories of groups

$\mathbb{G}$ is the category of all groups and group homomorphisms. Group homomorphisms are associative and closed under composition. Part (iv) of the definition follows by taking the identity automorphism of each group.

$\mathbb{TG}$ denotes the category of all topological groups under continuous (group) homomorphisms. The axioms of a category are easily verified in this case, using the observations made for the category $\mathbb{G}$.

Similarly, $\mathbb{HTG}$ is the category of Hausdorff topological groups under continuous group homomorphisms, $\mathbb{LCG}$ is the category of locally compact σ-compact groups under continuous group homomorphisms and $\mathbb{LIE}$ is the category of Lie groups under Lie group homomorphisms. It is easy to verify that these are indeed categories.

Example 5.1.3 The (small) category of a poset

Let $(P, \leq)$ be a partially ordered set. Then we can form a category $\mathbb{I}$ with $\text{ob}(\mathbb{I}) = P$ and given two elements $a, b \in P$ we define a unique morphism $\iota : a \to b$ if and only if $a \leq b$. It is important that the morphism $a \to b$ is unique when it exists to ensure that parts (ii)-(iv) of the definition hold. We should also note that part (iv) of the definition fails if the ordering is strict. One particular example which will be important is the category formed from the poset of all 1 and 2 element subsets of a finite set, under set inclusion.

Later, we may refer to a poset as a small category, implicitly using the association constructed in this example.

Definition 5.1.4 Diagrams

Let $\mathbb{I}$ be a small category. A diagram of $\mathbb{I}$ in a category $\mathbb{A}$ is a covariant functor $\delta : \mathbb{I} \to \mathbb{A}$. Specifically, this means that

$$\delta(f) : \delta(a) \to \delta(b) \in \mathrm{mor}(\mathbb{A}) \quad \text{whenever} \quad f : a \to b \in \mathrm{mor}(\mathbb{I}).$$

Example 5.1.5 A diagram in the category of groups

Let W be a Coxeter group with presentation

$$\left\langle S = \{s_1, s_2, \ldots, s_n\} \;\middle|\; \begin{array}{ll} (s_i s_j)^{m_{ij}} & \text{for all } i \neq j, \\ (s_i)^2 & \end{array} \right\rangle$$

and let $\mathbb{I}$ be the small category corresponding to the poset $(\mathcal{P}(S), \subseteq)$, where $\mathcal{P}(S)$ denotes the power set of S. There is a natural diagram $\delta : \mathbb{I} \to \mathbb{G}$ where

$$\delta(S') = \left\langle S' = \{s'_1, s'_2, \ldots, s'_k\} \subseteq S \;\middle|\; \begin{array}{ll} (s'_i s'_j)^{m_{ij}} & \text{for all } i \neq j, \\ (s'_i)^2 & \end{array} \right\rangle.$$

The set inclusion $S' \subseteq S''$ defines a monomorphism $\delta(S') \to \delta(S'')$. A proof of this can be approached from the viewpoint of Coxeter diagrams, as the subdiagram of W generated by the vertices of S' is itself a subdiagram of the Coxeter diagram generated by the vertices of S''.

Definition 5.1.6 Cones and colimits

Let $\delta : \mathbb{I} \to \mathbb{A}$ be a diagram. A cone over δ is a pair $(G, (\psi_i)_{i \in \mathrm{ob}(\mathbb{I})})$, where $G \in \mathrm{ob}(\mathbb{A})$ and $\psi_i : \delta(i) \to G$ are morphisms which satisfy

$$\psi_j \circ \delta(a) = \psi_i \quad \text{for all } i, j \in \mathrm{ob}(\mathbb{I}) \quad \text{and all } a \in \mathrm{mor}(i, j).$$

A cone $(G, (\psi_i)_{i \in \mathrm{ob}(\mathbb{I})})$ is called a colimit of δ if for each cone $(H, (\varphi_i)_{i \in \mathrm{ob}(\mathbb{I})})$ there is a unique morphism $\varphi : G \to H$ such that $\varphi \circ \psi_i = \varphi_i$.

Example 5.1.7 Completing example 5.1.5

By construction, we see that $(W, \iota_{S'})$ is a cone over the diagram δ, where $\iota_{S'} : \delta(S') \to W$ is the identity monomorphism. Moreover, given any other cone $(H, \varphi_{S'})$ there must be a monomorphism $\varphi : W \to H$ with the property that $\varphi|_{\iota_{S'}} = \varphi \circ \iota_{S'} = \varphi_{S'}$. Thus $(W, \iota_{S'})$ is a colimit of δ. The next proposition will prove that this colimit is unique up to isomorphism.

Proposition 5.1.8 Existence of colimits in categories of groups

Colimits exist for any diagram in the categories $\mathbb{G}$, $\mathbb{TG}$ and $\mathbb{HTG}$ and are unique up to isomorphism. That is, given any two colimits, $(G, (\psi_i)_{i \in \mathrm{ob}(\mathbb{I})})$ and $(H, (\varphi_i)_{i \in \mathrm{ob}(\mathbb{I})})$ of δ there is some (homeomorphic) isomorphism $\phi : G \to H$ such that $\phi \circ \psi_i = \phi^{-1}\varphi_i$ for all $i \in \mathrm{ob}(\mathbb{I})$.

Proof: Let $\delta : \mathbb{I} \to \mathbb{G}$ be a diagram. Then the group

$$G = \left. \bigsqcup_{i \in \mathrm{ob}(\mathbb{I})} \delta(i) \middle/ \sim \right.$$

is a colimit of δ, where $\sim$ is the equivalence relation $a \sim b$ if and only if there exist elements $i, j \in I$ such that $\{a, b\} \subseteq \delta(i) \cap \delta(j)$.

Using this result, a diagram $\delta : \mathbb{I} \to \mathbb{TG}$ clearly has a colimit $(G, (\lambda_i)_{i\in \mathrm{ob}(\mathbb{I})})$ in $\mathbb{G}$. There is a unique finest topology $\mathcal{O}$ on G which makes each λ_i continuous and $((G, \mathcal{O}), (\lambda_i)_{i\in \mathrm{ob}(\mathbb{I})})$ is a colimit of the diagram δ in $\mathbb{TG}$.

Similarly, a diagram $\delta : \mathbb{I} \to \mathbb{HTG}$ has a colimit $((G, \mathcal{O})(\lambda_i)_{i\in \mathrm{ob}(\mathbb{I})})$ in $\mathbb{TG}$. Let $H := G/\overline{\{1_G\}}$ and let $q : G \to G/\overline{\{1_G\}} = H$ be the canonical quotient homomorphism. Then $((H, \mathcal{O}')(q\circ\lambda_i)_{i\in \mathrm{ob}(\mathbb{I})})$ is a colimit of δ in $\mathbb{HTG}$, where $\mathcal{O}'$ is the quotient topology.

To prove uniqueness suppose (G, ψ_i) and (H, φ_i) are two colimits of $\delta : \mathbb{I} \to \mathbb{G}$. Then there exist maps $\psi : G \to H$ and $\varphi : H \to G$ such that

$$\varphi \circ \psi_i = \varphi_i \quad \text{and} \quad \psi \circ \varphi_i = \psi_i \quad \text{for all } i \in \mathrm{ob}(\mathbb{I}).$$

Therefore, $(\varphi \circ \psi) \circ \varphi_i = \varphi_i$ and $(\psi \circ \varphi) \circ \psi_i = \psi_i$ for all $i \in \mathrm{ob}(\mathbb{I})$. From this we deduce that $\varphi \circ \psi = \iota_G$ and $\psi \circ \varphi = \iota_H$, so $G \cong H$ and setting $\phi := \psi$ proves that the colimits are isomorphic. As the topologies chosen are the unique finest topologies it follows that colimits are unique in the other two categories as well. □

5.2 Final group topologies

Definition 5.2.1 Final group topologies

Let X be a set and let $(f_i)_{i\in I}$ be a collection of functions $f_i : X_i \to X$ where the X_i are topological spaces. The final topology on X is the finest topology under which each f_i is continuous. If X is a group then the topology is called the final group topology. An equivalent way of formulating this topology is to say that U is open in X if and only if $f_i^{-1}(U)$ is open in X_i for each $i \in I$.

Definition 5.2.2 Direct limit topologies

A directed set is a pair $(I, \leq)$ where I is a set and $\leq$ is a reflexive, transitive relation.

Let $\{A_i \,|\, i \in I\}$ be a family of topological spaces indexed by a directed set I and suppose there are continuous functions $f_{ij} : A_i \to A_j$ for all $i \leq j$ with the following properties:

(i) *f_{ii} is the identity homeomorphism on A_i,*

(ii) *$f_{ik} = f_{jk} \circ f_{ij}$.*

Then $(A_i, f_{ij})_{i,j\in I}$ is called a direct system over I.

The underlying set A of the direct limit, is given by

$$A = \varinjlim(A_i) := \left(\bigcup_i A_i\right) \Big/ \sim$$

where $x_i \sim x_j$ if and only if $x_i \in A_i$, $x_j \in A_j$ and $f_{ik}(x_i) = f_{jk}(x_j)$ for some $k \in I$.

The underlying set of a direct limit is constructed in much the same way as the colimit was constructed for δ in proposition 5.1.8.

Definition 5.2.3 Some topological properties

Let X be a topological space.

(i) *Let $C \subseteq X$. A point $x \in X$ is called an interior point of C if there is some open neighbourhood U of x contained in C.*

(ii) *X is called a Baire space if and only if any countable union of closed sets with empty interior has empty interior.*

(iii) *X is said to be σ-compact if and only if it is the union of countably many compact sets.*

Lemma 5.2.4 *Let G and H be Hausdorff topological groups and let $f : G \to H$ be a surjective continuous group homomorphism. If G is σ-compact and H is a Baire space, then f is an open map and H is locally compact.*

Proof: Since G is σ-compact it follows that $G = \bigcup_{n\in\mathbb{N}} K_n$ where each K_n is compact in G. By surjectivity and continuity, $H = \bigcup_{n\in\mathbb{N}} f(K_n)$ and each $f(K_n)$ is compact.

H must contain some interior point, so as H is Baire, $f(K_n)$ has nonempty interior for some n. H is therefore locally compact about this interior point, so as H is a topological group it must be locally compact about every point.

Let $\psi : G/\ker(f) \to H$ be the continuous group isomorphism induced by f and let $q : G \to G/\ker(f)$ be the quotient homomorphism. Then $\psi^{-1} \circ f = q$ which is continuous. Hence ψ is a homeomorphic isomorphism, so f is an open map. □

Proposition 5.2.5 *Let G be a Hausdorff topological group and let $(f_i)_{i\in I}$ be a countable family of continuous maps $f_i : X_i \to G$, such that X_i is σ-compact for each $i \in I$ and $\bigcup_{i\in I} f_i(X_i)$ generates G. Then G is locally compact if and only if it is a Baire space. In this case the given locally compact topology on G is the final group topology with respect to $(f_i)_{i\in I}$.*

Proof: Every locally compact space is a Baire space.

The final group topology $\mathcal{O}$ is at least as fine as the given topology so is Hausdorff. Since $(G, \mathcal{T})$ is generated by its σ-compact subset $\bigcup_{i\in I} f_i(X_i)$, the topological group $(G, \mathcal{T})$ is σ-compact. Also, as $(G, \mathcal{O})$ is Baire, lemma 5.2.4 shows us that the identity map $(G, \mathcal{T}) \to (G, \mathcal{O})$ is a homeomorphic isomorphism, so $(G, \mathcal{O})$ is locally compact. □

Corollary 5.2.6 *Let $\delta : \mathbb{I} \to \mathbb{LCG}$ be a diagram where $I = \mathrm{ob}(\mathbb{I})$ is countable. If there exists a locally compact Hausdorff group topology $\mathcal{O}$ on G making $\lambda_i : G_i \to (G, \mathcal{O})$ continuous for each $i \in I := \mathrm{ob}(\mathbb{I})$ and such that $(G, (\lambda_i)_{i\in I})$ is a colimit of δ in $\mathbb{G}$. Then $((G, \mathcal{O}), (\lambda_i)_{i\in I})$ is a colimit of δ in the categories of topological groups, Hausdorff topological groups and locally compact groups. Furthermore, if each G_i is a σ-compact Lie group and $(G, \mathcal{O})$ is a Lie group then $((G, \mathcal{O}), (\lambda_i)_{i\in I})$ is a colimit of δ in the category $\mathbb{LIE}$ of Lie groups under Lie group homomorphisms.*

Proof: By proposition 5.2.5 we know that $((G, \mathcal{O}), (\lambda_i)_{i\in \mathrm{ob}(\mathbb{I})})$ is a colimit of δ in the category of topological groups, so as this group is also Hausdorff and locally compact it follows that it is the colimit in these categories as well. Finally, if all these groups are Lie groups, then this is also a colimit in $\mathbb{LIE}$.

5.3 A topological version of Borovoi's theorem

Let K be a simply connected compact semisimple Lie group, with lie algebra $\mathfrak{g}_0$ and let T_K be a maximal torus of K with Lie algebra $\mathfrak{t}_0$. Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g} = (\mathfrak{g}_0)^{\mathbb{C}}$ and let T be a maximal torus of G containing T_K. Let $\Phi = \Phi(G, T)$ be the corresponding root system and let Π be a system of simple roots of Φ.

To each root $a \in \Pi$ we may associate some compact semisimple Lie group $K_a \cong SU_2(\mathbb{C})$ of rank one such that T normalises K_a. For any two distinct simple roots $a, b \in \Pi$ we denote by K_{ab} the group generated by K_a and K_b and by Φ_{ab} we denote its root system relative to the torus $T_{ab} = T \cap K_{ab}$. K_{ab} is a compact semisimple Lie group of rank two and $\{a, b\}$ is a simple system for Φ_{ab}.

The pair (K_a, K_b) is called a standard pair of K_{ab} as is any image of (K_a, K_b) under a homomorphism from K_{ab} onto a central quotient of K_{ab}. Standard pairs are conjugate in G, a result which can be determined using the fact that maximal tori are conjugate (3.3.5).

Definition 5.3.1 Fundamental domains

Let A be a poset and let G be a group which acts on A. $F \subseteq A$ is called a fundamental domain for the action of G on A if the following hold.

(i) *$b \in F$ whenever $a \in F$ and $b \leq a$,*

(ii) $A = G(F) = \{g(a) \mid a \in F, g \in G\}$,

(iii) $G(a) \cap F = \{a\}$ *for all* $a \in F$.

In particular, if G is a group acting strongly transitively on its associated thick building $\Delta \cong G/B$, and σ is the chamber corresponding to the chosen subgroup B of G, then $\Delta_{\leq\sigma}$ is a fundamental domain for the action of G on Δ.

Lemma 5.3.2 Tits' lemma

Let A be a poset, let G be a group which acts on A and let F be a fundamental domain for the action of G on A. Let $\mathbb{I}$ be a small category with $\mathrm{ob}(\mathbb{I}) = F$ and morphisms given by $a \to b$ whenever $b \leq a \in F$. Then the poset A is simply connected if and only if G is the colimit of the diagram $\delta : \mathbb{I} \to \mathbb{G}$, where $\delta(a) = G_a$ and $\delta(a \to b) = (G_a \hookrightarrow G_b)$. (A poset is simply connected if and only if the associated simplicial complex is).

Proof: See [Ti86]. □

Theorem 5.3.3 Topological version of Borovoi's theorem

Let K be a simply connected compact semisimple Lie group with topology $\mathcal{O}$, let T_K be a maximal torus of K, let $\Phi = \Phi(G,T)$ be the corresponding root system and let Π be a simple system for Φ. Let $\mathbb{I}$ be a small category whose objects are the one and two element subgroups of Π. This is often written as $\mathrm{ob}(\mathbb{I}) = \binom{\Pi}{1} \cup \binom{\Pi}{2}$. The morphisms of $\mathbb{I}$ are precisely the inclusions $\{a\} \to \{a,b\}$ for all $a,b \in \Pi$. Let $\delta : \mathbb{I} \to \mathbb{LCG}$ be a diagram with $\delta(\{a\}) = K_a, \delta(\{a,b\}) = K_{ab}$ and $\delta(\{a\} \to \{a,b\}) = K_a \hookrightarrow K_{ab}$.

Then $\left((K,\mathcal{O}), (\iota_i)_{\binom{\Pi}{1}\cup\binom{\Pi}{2}}\right)$ is a colimit of δ in the category $\mathbb{LIE}$ of Lie groups.

Proof: If the rank of K is at most 2 the result holds trivially, so we may assume that $|\Pi| \geq 3$. Let B be a Borel subgroup containing the maximal torus T of G and construct the associated building $\Delta \cong G/B$. The set $\Delta_{\leq\sigma}$ where σ is the chamber of Δ corresponding to B is a fundamental domain for the conjugation action of G on Δ. Since K is a compact real form of G, the Iwasawa decomposition tells us that $G = KB$ so $\Delta_{\leq\sigma}$ is also a fundamental domain for the conjugation action of K on Δ.

The stabilisers in K of the subsimplices of codimensions 1 and 2 are precisely the groups K_aT_K and $K_{ab}T_K$ respectively. Tits' lemma and the Solomon-Tits' theorem (4.3.4) together allow us to deduce that K is the colimit of the diagram

$$\delta : \mathbb{I} \to \mathbb{G}, \quad \text{where} \quad \delta(a) = \mathrm{Stab}_K(a) \quad \text{and} \quad \delta(a \to b) = (\mathrm{Stab}_K(a) \hookrightarrow \mathrm{Stab}_K(b)).$$

This can be done as buildings of rank at least three are simply connected. To get a result concerning the groups K_aT_K and $K_{ab}T_K$ we must apply induction on $|\Pi|$, taking stabilisers over more and more roots, until we reach the stabilisers of subsimplices of codimension 1 and 2. We can do this as there are exactly two elements of Π remaining when we obtain the groups K_aT_K and $K_{ab}T_K$, so at all previous times (when we may be hoping to apply Tits' lemma) there were at least three, so the rank of the associated building was at least three and therefore simply connected.

From this we deduce that the group K is a colimit in the category of abstract groups of the diagram

$$\delta' : \mathbb{I} \to \mathbb{LCG} \quad \text{with} \quad \delta'(\{a\}) = K_aT_K, \quad \delta'(\{a,b\}) = K_{ab}T_K \quad \text{and} \quad \delta'(\{a\} \to \{a,b\}) = (K_aT_K \hookrightarrow K_{ab}T_K).$$

This is still some way from what we need to prove. However, it is known that the torus T_K can be reconstructed from the rank two tori T_{ab} (see [GLS95], lemma 29.3) so G is in fact a colimit of the diagram δ in the category of abstract groups. Applying corollary 5.2.6 to δ completes the theorem. □

Chapter 6

Locally k_ω topological groups

k_ω spaces play an important role in the theory of Kac-Moody groups as it will be shown that a Kac-Moody group, endowed with the Kac-Peterson topology is a k_ω topological group. The information contained in this chapter will be used to prove this result and allow us to deduce that the category $\mathbb{KOG}$ of k_ω groups and continuous homomorphisms is the natural setting in which to attempt any extension of Borovoi's theorem to compact real forms of complex Kac-Moody groups.

6.1 Locally k_ω topological spaces

Definition 6.1.1 **k_ω spaces**

Let X be a Hausdorff topological space. X is called a k_ω space if there exists a sequence of compact subsets

$$K_1 \subseteq K_2 \subseteq \cdots \subseteq X$$

such that $X = \bigcup_{n\in\mathbb{N}} K_n$ and $U \subseteq X$ is open if and only if $U \cap K_n$ is open in K_n (with the subspace topology) for each $n \in \mathbb{N}$. The sequence (K_n) is called a k_ω sequence for X.

A Hausdorff topological space X is said to be locally k_ω if every point has an open neighbourhood which is k_ω in the subspace topology.

Example 6.1.2 **Examples of k_ω and locally k_ω spaces**

(i) Every k_ω space is locally k_ω.

(ii) Every compact Hausdorff topological space is a k_ω space.

(iii) Any countable discrete topological space is a k_ω space and any discrete topological space is locally k_ω.

(iv) $\mathbb{R}^n$ is a noncompact k_ω space with k_ω sequence $([-k,k]^n)_{k\in\mathbb{N}}$.

Lemma 6.1.3 *Every quotient map $f : X \to Y$ between k_ω spaces is compact covering. In other words, for each compact set $K \subseteq Y$ there is a compact set $L \subseteq X$ with the property that $K \subseteq f(L)$.*

Proof: Follows from [Gl05], lemma 1.7(d). □

Proposition 6.1.4 **More k_ω spaces**

(i) *Locally compact Hausdorff spaces are locally k_ω.*

(ii) *σ-compact, locally compact Hausdorff spaces are k_ω.*

(iii) *Closed subsets of k_ω spaces are k_ω in the subspace topology.*

(iv) *Finite products of k_ω spaces are k_ω in the product topology.*

(v) *Countable disjoint unions of k_ω spaces are k_ω.*

(vi) *Hausdorff quotients of k_ω spaces are k_ω.*

(vii) *Open subsets of locally k_ω spaces are locally k_ω in the subspace topology.*

(viii) *Closed subsets of locally k_ω spaces are locally k_ω in the subspace topology.*

(ix) *Finite products of locally k_ω spaces are locally k_ω in the product topology.*

(x) *Hausdorff quotients of locally k_ω spaces under open quotient maps are locally k_ω.*

Proof:

(i) Each point is contained in an open neighbourhood which is compact in the subspace topology and therefore is k_ω. As the space is Hausdorff it is locally k_ω.

(ii) The space is locally k_ω as it is locally compact and is covered by countably many compact sets. Using these facts we can build a k_ω sequence for the space. Thus the space is k_ω, since we assumed it was Hausdorff.

(iii) Intersecting the closed subset with each of the terms in the k_ω sequence yields a k_ω sequence for the closed subspace.

(iv) Create a k_ω sequence for the product space by taking the pointwise direct product of the k_ω sequences.

(v) To find a k_ω sequence for this space we define $(K_n^j)_{n\in\mathbb{N}}$ to be a k_ω sequence of the jth element of the disjoint union. We then form a k_ω sequence (K_n) for the disjoint union of these spaces via the rule
$$K_n = \coprod_{i=1}^{n} K_n^i.$$

(vi) The quotient map is compact covering, by lemma 6.1.3, so the image of a k_ω sequence can be enlarged to a k_ω sequence in the quotient space. The enlarging must be carefully chosen to ensure that all open sets are open in each subspace, but this can be accomplished. The underlying space is also Hausdorff, by assumption, so is k_ω.

For a proof of part (vii) see [GGH09] page 11. The other locally k_ω properties follow relatively easily from the corresponding k_ω ones. □

We will use these properties of k_ω spaces repeatedly. Now we present one additional lemma which will be needed for the propositions which complete this introductory section.

Lemma 6.1.5 *Let X be locally k_ω and let $Y \subseteq X$ be σ-compact. Then Y has an open neighbourhood $U \subseteq X$ which is a k_ω space.*

Proof: Y is σ-compact so there exists a sequence (U_n) of open $k\omega$-spaces contained in X such that
$$Y \subseteq U := \bigcup_{n\in\mathbb{N}} U_n.$$
U is a Hausdorff quotient of the k_ω space $\coprod_{n\in\mathbb{N}} U_n$ so is itself a $k\omega$-space, by proposition 6.1.4 (v) and (vi). □

The final two results from this section prove that k_ω spaces are a excellent setting in which to study direct limits, a result which will be crucial when extending Borovoi's theorem.

Proposition 6.1.6 *Let $X_1 \subseteq X_2 \subseteq \dots$ be a sequence of k_ω spaces such that the inclusion map $X_n \hookrightarrow X_{n+1}$ is continuous for each $n \in \mathbb{N}$. Then the final topology $\mathcal{O}$ on $X := \bigcup_{n\in\mathbb{N}} X_n$ with respect to the inclusion maps $X_n \hookrightarrow X$ is k_ω and makes $(X, \mathcal{O})$ the direct limit of the X_n in the categories of topological spaces, Hausdorff topological spaces and k_ω spaces.*

This proposition also holds if the condition that X and X_n are k_ω spaces is replaced by them being locally k_ω spaces and the conclusion that $(X, \mathcal{O})$ is a direct limit in the category of $k\omega$ spaces is replaced by that of locally k_ω spaces.

Proof: It suffices to show that $(X, \mathcal{O})$ is k_ω and is the direct limit in the category of topological spaces. The rest then follows trivially.

As each X_j is a k_ω space, let $(K_n^j)_{n\in\mathbb{N}}$ be a k_ω sequence for X_j. Replacing K_n^j by $\bigcup_{i=1}^j K_n^i$ wherever necessary we may assume that $K_n^i \subseteq K_n^j$ whenever $i \leq j$.

Define $K_n := K_n^n$ for all $n \in \mathbb{N}$. Each K_n is compact and the inclusions $K_n \hookrightarrow K_{n+1}$ are topological embeddings, so using [Gl03] proposition 3.6(a) we can see that the final topology $\mathcal{T}$ on X with respect to the inclusion maps $K_n \hookrightarrow X$ is Hausdorff and therefore $(X, \mathcal{T})$ is a k_ω space as (K_n) forms a k_ω sequence for X. It now remains to show that $\mathcal{T} = \mathcal{O}$.

As the inclusion maps $K_n \hookrightarrow X_a \hookrightarrow (X, \mathcal{O})$ are continuous, $\mathcal{O} \subseteq \mathcal{T}$ by the definition of a final topology (5.2.1). Now suppose $U \in \mathcal{T}$. $U \cap K_n$ is open in K_n for all $n \in \mathbb{N}$. Define $k := \max\{j, n\}$ and note that $U \cap K_n^j = (U \cap K_m) \cap K_n^j$ which is open in K_n^j and the inclusion map $K_n^j \hookrightarrow K_m$ is continuous. Hence $U \cap X_j$ is open in X_j for each j and $U \in \mathcal{O}$, as required. $\square$

For the locally k_ω case, see [GGH09] page 12.

From this proposition we may deduce that the direct limit space of each countable direct system of k_ω spaces is itself a k_ω space.

Proposition 6.1.7 *Let $X_1 \subseteq X_2 \subseteq \dots$ and $Y_1 \subseteq Y_2 \subseteq \dots$ be sequences of topological spaces with continuous inclusion maps.*

Set $X := \bigcup_{n\in\mathbb{N}} X_n = \varinjlim(X_n)$ and $Y := \bigcup_{n\in\mathbb{N}} Y_n = \varinjlim(Y_n)$ endowed with the direct limit topology.

Denote by $\varinjlim(X_n \times Y_n)$ the set $\bigcup_{n\in\mathbb{N}}(X_n \times Y_n)$ equipped with the direct limit topology over the inclusion maps $X_n \times Y_n \hookrightarrow X \times Y$.

The natural map

$$\beta\colon \varinjlim(X_n \times Y_n) \to \varinjlim(X_n) \times \varinjlim(Y_n) \quad \text{given by } (x, y) \to (x, y)$$

is a continuous bijection. If each X_n and Y_n is a locally k_ω space, then β is a homeomorphism.

Proof: The inclusion $X_n \times Y_n \hookrightarrow X \times Y$ is continuous for each $n \in \mathbb{N}$ by assumption, so β is continuous. It is also a bijection as the operations of taking direct limits and products commute in the category of sets.

To prove the second part we will show that for each pair $(x, y) \in X \times Y$ there are open neighbourhoods $U \subseteq X$ of X and $V \subseteq Y$ of y such that $X \times Y$ and $\varinjlim(X_n \times Y_n)$ induce the same topology on $U \times V$. Without loss of generality we may assume $(x, y) \in X_1 \times Y_1$.

Each X_n and Y_n is locally k_ω, so we let U_1 and V_1 be open k_ω neighbourhoods of x in X_1 and y in Y_1 respectively. Using lemma 6.1.5 we construct non-descending (by inclusion) sequences of open k_ω spaces (U_n) and (V_n) where each $U_n \subseteq X$ and each $V_n \subseteq Y$. Define

$$U := \bigcup_{n\in\mathbb{N}} U_n \quad \text{and} \quad V := \bigcup_{n\in\mathbb{N}} V_n.$$

By lemma 6.1.4, parts (v) and (vi) as applied in the proof of lemma 6.1.5, U and V are both k_ω spaces with respect to the topology induced by X and Y respectively. We define k_ω sequences (K_n^j) for U_j and (L_n^j) for V_j, as was done in the proof of proposition 6.1.6 and obtain k_ω sequences (K_n) for U and (L_n) for V by setting $K_n = K_n^n$ and $L_n = L_n^n$.

Proposition 3.3 of [Gl03] shows that direct products and countable direct limits behave well on compact spaces, so we may deduce that

$$U_j \times V_j = \varinjlim(K_n^j) \times \varinjlim(L_n^j) = \varinjlim(K_n^j \times L_n^j).$$

Therefore $(K_n^j \times L_n^j)$ is a k_ω sequence for $U_j \times V_j$ and it follows that $(K_n \times L_n)$ is a k_ω sequence for the open subset $\varinjlim(U_n \times V_n)$ of $\varinjlim(X_n \times Y_n)$ equipped with the subspace topology.

Using proposition 3.3 of [Gl03] again we see that

$$U \times V = \varinjlim(K_n) \times \varinjlim(L_n) = \varinjlim(K_n \times L_n) = \varinjlim(U_n \times V_n).$$

$\varinjlim(U_n \times V_n)$ is simply the set $U \times V$ equipped with the subspace topology induced from $\varinjlim(X_n \times Y_n)$. (For a more detailed discussion of this fact, see [Gl03] lemma 3.1). Thus $X \times Y$ and $\varinjlim(X_n \times Y_n)$ induce the same topology on $U \times V$, so the two spaces are locally homeomorphic about each point. Thus β is a homeomorphism. □

6.2 Locally k_ω groups

Definition 6.2.1 **k_ω and locally k_ω groups**

A topological group is said to be a k_ω group if it is also a k_ω space. Identically, a locally k_ω group is a topological group which is also a locally k_ω space.

Proposition 6.2.2 *The following conditions are equivalent on a topological group G.*

(i) *G is a locally k_ω group,*

(ii) *G has an open subgroup H which is a k_ω group,*

(iii) *$G = \varinjlim(X_n)$ as a topological space for some non-descending sequence $X_1 \subseteq X_2 \subseteq \dots$ of closed locally compact subsets X_n of G equipped with the induced topology.*

Proof: See [GGH09] proposition 5.3. □

Proposition 6.2.3 *Let $G_1 \subseteq G_2 \subseteq \dots$ be a non-descending sequence of k_ω groups such that the inclusion maps $G_n \hookrightarrow G_{n+1}$ are continuous homomorphisms. $G := \bigcup_{n\in\mathbb{N}} G_n$ permits a group structure which makes each inclusion $G_n \hookrightarrow G$ a homomorphism.*

The final topology with respect to these inclusion maps makes G a k_ω group and also the direct limit $\varinjlim(G_n)$ in the categories of topological spaces and topological groups.

Proof: G is trivially the direct limit in the category of topological spaces and by proposition 6.1.6, G is a k_ω group when equipped with this topology. It remains to show that G is a topological group.

Let $\lambda_n : G_n \to G_n$ be the inversion map on G_n and $\lambda : G \to G$ the inversion map on G. Then $\lambda = \varinjlim(\lambda_n)$ and λ is continuous as each G_n is continuous. The direct limit map $\lambda = \varinjlim \lambda_n$ is the map from $G \to G$ which extends each of the maps $\lambda_n : G_n \to G_n$, it is well defined as $G_n \subseteq G$, G_n is closed under taking inverses and the inverse of an element in G_n is the same as the inverse of that element in G_m for all $m \geq n$. The corresponding map for group multiplication is also shown to be well-defined by similar observations.

Let $\mu_n : G_n \times G_n \to G_n$ be the group multiplication map on G_n and $\mu : G \times G \to G$ the group multiplication map on G. Identifying the topological spaces $G \times G$ with $\varinjlim(G_n \times G_n)$ by applying proposition 6.1.7, μ is given by $\varinjlim(\mu_n) : \varinjlim(G_n \times G_n) \to \varinjlim(G_n)$ which is continuous. Therefore G is a topological group. □

An important remark concerning the previous theorem is that it also holds for locally k_ω spaces, as both propositions 6.1.6 and 6.1.7 have analogues for these spaces. We now give an important corollary which relates more specifically to the setting of amalgams of Kac-Moody groups.

Corollary 6.2.4 *Let I be a countable directed set (see definition 5.2.2) and let $\mathcal{S} := ((G_i)_{i\in I}, (f_{ij})_{i\geq j})$ be a direct system of k_ω groups and continuous homomorphisms. Then $\mathcal{S}$ has a direct limit $(G, f_i)_{i\in I}$ in the category of Hausdorff topological groups. The topology on G is the final group topology with respect to the family (f_i) and the Hausdorff topological space G is the direct limit of $\mathcal{S}$ in the category of Hausdorff topological spaces. Furthermore, G is a k_ω group and thus is a direct limit of $\mathcal{S}$ in the category of k_ω groups as well.*

Proof: Without loss of generality we may assume that $I = (\mathbb{N}, \leq)$. Set $N_m := \overline{\bigcup_{n \geq m} \ker(f_{n,m})}$, where the bar notation denotes the topological closure. Set $Q_m := G_m/N_m$ and let $q_m : G_m \to Q_m$ be the respective quotient morphism. Proposition 6.1.4(v) tells us that each Q_m is a k_ω group.

Let $g_{n,m} : Q_m \to Q_n$ be the continuous homomorphism defined by the relation $g_{n,m} \circ q_m = q_n \circ f_{n,m}$. Therefore, the direct limit of $\mathcal{S}$ in the category of Hausdorff topological spaces is precisely the direct limit of $((Q_n)_{n\in\mathbb{N}}, (g_{n,m})_{n\geq m})$.
By proposition 6.2.3, the direct limit of the latter system is a k_ω group. □

Proposition 6.2.5 *Let G be a group and let $(f_i)_{i\in I}$ be a countable family of maps $f_i : X_i \to G$ such that each X_i is a k_ω space and $\bigcup_{i\in I} X_i$ generates G. If the final group topology on G with respect to the family (f_i) is Hausdorff, then G is a k_ω group with respect to this topology.*

The proof of this proposition requires knowledge of the Markov free topological group and therefore is omitted. [Ma50] and [MMO73] provide the details necessary to complete a proof. □

Unlike the previous results, we do not claim that an analogue of this result holds for locally k_ω groups.

Corollary 6.2.6 *Let $\delta : \mathbb{I} \to \mathbb{KOG}$ be a diagram in the category $\mathbb{KOG}$ of k_ω groups under continuous homomorphisms, where $I := \mathrm{ob}(\mathbb{I})$ is countable. Write $G_i := \delta(i)$ for all $i \in \mathrm{ob}(\mathbb{I})$ and $\psi_a := \delta(a) : G_i \to G_j$ for all $a \in \mathrm{mor}(i,j)$. Let $(G, (\lambda_i)_{i\in I})$ be a colimit of δ in the category of Hausdorff topological groups. Then G is a k_ω group and $(G, (\lambda_i)_{i\in I})$ is a colimit of δ in the category of k_ω groups.*

Proof: This follows directly from proposition 6.2.5. □

This corollary is an analogue of corollary 5.2.6 in the category $\mathbb{KOG}$. We will see that it also plays a significant role when extending Borovoi's theorem.

With this introduction to k_ω groups complete we proceed to the study of Kac-Moody groups, where the motivation for this chapter will become clear.

Chapter 7

Kac-Moody groups

7.1 Kac-Moody algebras

We will introduce Kac-Moody algebras and show that they are in some senses a natural generalisation of complex semisimple Lie algebras to include infite dimensional algebras. The theory will be illustrated by the key example $\mathfrak{sl}_n(\mathbb{C}[t,t^{-1}])$ where $\mathbb{C}[t,t^{-1}]$ is the infinite dimensional algebra of Laurent polynomials with complex coefficients. The main sources for this chapter are [Ka85] and [GGH09], section 6.

Definition 7.1.1 Generalised Cartan matrices

An $n \times n$ square matrix $A = (a_{ij})$ is said to be a generalised Cartan matrix if $a_{ii} = 2$ and for each $i, j \in \{1, \ldots, n\}$ with $i \neq j$

(i) $a_{ij} \in \mathbb{Z}_0^- = \{z \in \mathbb{Z} \,|\, z \leq 0\}$,

(ii) $a_{ij} = 0$ *if and only if* $a_{ji} = 0$.

We can see from chapter 2 that all abstract Cartan matrices are generalised Cartan matrices, justifying the term 'generalised'.

Given a generalised Cartan matrix A we can define a Dynkin diagram Γ_A for this matrix in exactly the same way as for an abstract Cartan matrix with the one additional condition that if $A_{ij}A_{ji} \geq 4$ then we label the edge between the ith and jth vertices $+\infty$. This project will only consider 2-spherical generalised Cartan matrices, that is, $n \times n$ generalised Cartan matrices whose 2×2 submatrices of the form

$$\begin{pmatrix} A_{ii} & A_{ij} \\ A_{ji} & A_{ij} \end{pmatrix}$$

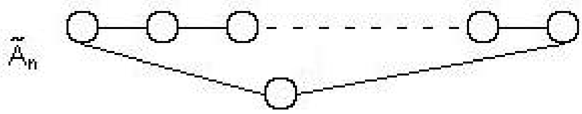
for $1 \leq i < j \leq n$ are abstract Cartan matrices in the sense of definition 2.4.7. In terms of Dynkin diagrams, this means that the restriction of Γ_A to any two vertices is isomorphic to $A_1 \oplus A_1$, A_2, B_2 or G_2, so in particular $A_{ij}A_{ji} \leq 3$ for all $i \neq j$.

Example 7.1.2 A 2-spherical generalised Cartan matrix

The matrix
$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & & & 0 \\ 0 & -1 & 2 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & -1 & 0 \\ 0 & & & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

is a generalised Cartan matrix but not an abstract Cartan matrix, as the Dynkin diagram it generates is not one of those from figure 2.1.2.
We deduce from this that A fails property (v) of definition 2.4.7.

The Dynkin diagram of A, is called $\widetilde{A_n}$. It has $(n+1)$ vertices and is of the form



Definition 7.1.3 Realisations of generalised Cartan matrices

A realisation of an $n \times n$ generalised Cartan matrix A of rank l is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where $\mathfrak{h}$ is a vector space over $\mathbb{C}$, $\Pi = \{c_1, c_2, \ldots, c_n\} \subseteq \mathfrak{h}^*$ and $\Pi^\vee = \{h_1, h_2, \ldots, h_n\} \subseteq \mathfrak{h}$, which satisfies the following properties.

(i) Π *and* $\Pi^\vee$ *are linearly independent in* $\mathfrak{h}^*$ *and* $\mathfrak{h}$ *respectively,*

(ii) $c_i(h_j) = a_{ij}$ *for all* i *and* j,

(iii) $\dim(\mathfrak{h}) = 2n - l$.

We say two realisations $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ are isomorphic if there is a vector space isomorphism $\psi : \mathfrak{h}_1 \to \mathfrak{h}_2$ such that $\psi^*(\Pi_1) = \Pi_2$ and $\psi(\Pi_1^\vee) = \Pi_2^\vee$, where ψ^* is the dual of ψ.

Proposition 7.1.4 *Let A be an $n \times n$ generalised Cartan matrix. A has a unique realisation up to isomorphism. Moreover, the realisations of two matrices A and B are isomorphic if and only if A can be obtained from B by a permutation of the index set $\{1, \ldots, n\}$.*

Proof: See [Ka85] proposition 1.1. □

Using this proposition we may define the Kac-Moody algebra corresponding to a generalised Cartan matrix A.

Definition 7.1.5 The derived Lie algebra $\mathfrak{g}'(A)$

Let $A = (a_{ij})$ be an $n \times n$ generalised Cartan matrix. The derived Lie algebra of type A, denoted $\mathfrak{g}'(A)$, is the Lie algebra generated by the set of elements $\{e_i, f_i, h_i \,|\, 1 \le i \le n\}$ subject to the relations:

(i) $[h_i, e_j] = A_{ij} e_i$,

(ii) $[h_i, f_j] = -A_{ij} f_j$,

(iii) $[h_i, h_j] = 0$,

(iv) $[e_i, f_j] = -\delta_{ij} h_i$.

We note here that the h_i's can only generate $\mathfrak{h}$ as a vector space if the generalised Cartan matrix has rank n. However, the set $\{e_i, f_i, h_i \,|\, 1 \le i \le n\}$ generates the same Lie algebra as that generated by $\mathfrak{h} \cup \{e_i, f_i \,|\, 1 \le i \le n\}$.
By defining the bracket to be antisymmetric, we can see that $\mathfrak{g}'(A)$ is indeed a Lie algebra. Using the results from chapter 2 we can deduce that $\mathfrak{g}'(A)$ is finite dimensional if and only if A is an abstract Cartan matrix as defined by proposition 2.4.7.

$\mathfrak{h}$ is a maximal simultaneously diagonalisable abelian subalgebra of $\mathfrak{g}'(A)$ so is a Cartan subalgebra of $\mathfrak{g}'(A)$. (In the infinite dimensional case, we require that $\mathfrak{h}$ is maximal amongst the simultaneously diagonalisable abelian Lie subalgebras of $\mathfrak{g}$. Corollary **??**(i) shows that such a condition is unnecessary in the finite-dimensional case.)

Using the method discussed immediately after definition 2.3.1, we can define a root-space decomposition of $\mathfrak{g}'(A)$ relative to $\mathrm{ad}(\mathfrak{h})$.

$$\mathfrak{g}'(A) = \bigoplus_{a \in \mathfrak{h}^*} \mathfrak{g}_a = \mathfrak{h} \oplus \bigoplus_{a \in \Phi} \mathfrak{g}_a.$$

where $\mathfrak{g}_a = \{X \in \mathfrak{g} \,|\, [H, X] = a(H)X \text{ for all } H \in \mathfrak{h}\}$. Notice that $\mathfrak{h} = \mathfrak{g}_0$.

We will use the same terminology that was used in chapter 2, so for $a \neq 0$, $\mathfrak{g}_a$ is called a root-space if it is nontrivial and in such cases a is called a root. Elements of $\mathfrak{g}_a$ are called root vectors of the root a. The set of roots is denoted by Φ.

Returning to the definition of a realisation, we note that Π is always a subset of Φ, explaining why Π is sometimes referred to as the root basis.

Again the Weyl group is the group W generated by the root reflections $S := \{s_a \mid a \in \Pi\}$. Unlike in the finite dimensional case, we cannot guarantee that $W\Pi = \Phi$. To distinguish this we partition $\Phi = \Phi_R \cup \Phi_I$ where $\Phi_R := W\Pi$.
Φ_R is called the set of real roots and Φ_I the set of imaginary roots.

Proposition 7.1.6 An important ideal of the derived Lie algebra

The collection of all ideals of $\mathfrak{g}'(A)$ which intersect $\mathfrak{h}$ trivially has a unique maximal element, $\mathfrak{a}$.

Definition 7.1.7 The Kac-Moody algebra $\mathfrak{g}(A)$

Let A be a generalised Cartan matrix with derived Lie algebra $\mathfrak{g}'(A)$. Let $\mathfrak{a}$ be the ideal guaranteed by proposition 7.1.6. Then the Kac-Moody algebra is defined as

$$\mathfrak{g}(A) = \mathfrak{g}'(A)/\mathfrak{a}.$$

The root-space decomposition on $\mathfrak{g}'(A)$ induces a decomposition on $\mathfrak{g}(A)$.

We will outline an alternative construction of the Kac-Moody algebra $\mathfrak{g}(A)$ in the case where A is symmetrisable, so $A = DS$ where D is an invertible diagonal matrix and S is a symmetric matrix. In this case we can obtain the Kac-Moody algebra directly from definition 7.1.7 by imposing the two additional conditions

$$(\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) = 0 = (\mathrm{ad}_{f_i})^{1-a_{ij}}(f_j).$$

The Gabber-Kac theorem [Ka85](9.11) shows that these two constructions give equivalent Kac-Moody algebras.

Example 7.1.8 $\mathfrak{sl}_n(\mathbb{C}[t, t^{-1}])$, for $n \geq 3$

Let $\mathfrak{g} := \mathfrak{sl}_n(\mathbb{C}[t, t^{-1}]) := \{X \in M_n(\mathbb{C}[t, t^{-1}]) \mid \mathrm{Tr}(X) = 0\}$, where $\mathbb{C}[t, t^{-1}]$ is the integral domain of all complex valued Laurent polynomials of the form

$$\sum_{|k| \leq N} c_k t^k \text{ for some } N.$$

We wish to show that for $n \geq 3$, $\mathfrak{g}$ is in fact isomorphic to the Kac-Moody algebra $\mathfrak{g}(A)$ where $A = (A_{ij})$ is the $n-1 \times n-1$ generalised Cartan matrix given in example 7.1.2.

Firstly, we will form a set of generators contained in $\mathfrak{g}$ for the derived Lie algebra $\mathfrak{g}'(A)$.
For each $i \in \mathbb{Z}_n = \{1, \ldots, n\}$, let e_i be the matrix which is 0 in all but the $(i+1, i)$ entry, where it takes value t. Similarly, let f_i be the matrix which is 0 in all but the $(i, i+1)$ entry, where it takes value $-t^{-1}$. We also define h_i to be zero except in the ith diagonal entry where it takes value 1 and in the $(i+1)$th diagonal entry where it takes value (-1).

An elementary, but time consuming, calculation shows that $\mathfrak{g}$ is in fact generated by the set

$$\{e_i, f_i, h_i \mid 1 \leq i \leq n\} \quad \text{as a Lie algebra.}$$

We now need to verify that the four rules in definition 7.1.5 hold.

(i) $[h_i, e_i] = h_i e_i - e_i h_i = e_i - (-e_i) = 2e_i = A_{ii} e_i$, also

$[h_i, e_{i+1}] = -e_{i+1} - 0 = A_{i,i+1} e_{i+1}$ and $[h_i, e_{i-1}] = 0 - e_{i-1} = A_{i,i-1} e_{i-1}$.

The remaining case holds trivially.

(ii) $[h_i, f_i] = h_i f_i - f_i h_i = -f_i - (f_i) = -2f_i = -A_{ii} f_i$, similarly, $[h_i, f_{i\pm1}] = f_{i\pm1} = -A_{i,i\pm1} f_i$ and the final case is also simple.

(iii) The span of $\{h_i \mid 1 \leq i \leq n\}$ is an abelian subset of $\mathfrak{g}$, so this part holds.

(iv) $[e_i, f_i] = e_i f_i - f_i e_i$. This matrix takes value $t(-t^{-1}) = -1$ on the ith diagonal entry and value $-(-t^{-1})t = 1$ on the $(i+1)$th diagonal entry. Therefore $[e_i, f_i] = -h_i = -\delta_{ij} h_i$.

If $i \neq j$, then $e_i f_j = f_j e_i = 0$, so $[e_i, f_j] = 0 = -\delta_{ij} h_i$.

This proves that $\mathfrak{g}$ is a quotient of the derived Lie algebra $\mathfrak{g}'(A)$, however, we can also show that this construction obeys the two additional conditions $(\mathrm{ad}_{e_i})^{1-A_{ij}}(e_j) = 0 = (\mathrm{ad}_{f_i})^{1-A_{ij}}(f_j)$, so using the fact that A is symmetric and the Gabber-Kac theorem again, we can deduce that $\mathfrak{g}$ is a quotient of the Kac-Moody algebra $\mathfrak{g}(A)$.

If $i = j$ there is nothing to prove, so we assume $i \neq j$.

(i) For $j = i \pm 1$, $(\mathrm{ad}_{e_i})^{1-A_{ij}}(e_j) = [e_i, [e_i, e_j]] = 0$. To see this we can restrict our attention to 3×3 submatrices obtained by deleting all but the ith, jth, $(i+1)$th and $(j+1)$th rows and columns. From this we see that $[e_i, e_j]$ takes value $\pm t^2$ in the top right entry of this 3×3 matrix and is zero everywhere else.
Therefore, $e_i[e_i, e_j] = [e_i, e_j]e_i = 0$ and the result follows.

(ii) For $j \notin \{i-1, i, i+1\}$, $[e_i, e_j] = 0$, so $[e_i, [e_i, e_j]] = [e_i, 0] = 0$.

(iii) The case for f_i is constructed similarly, with the 3×3 matrix obtained from $[f_i, f_{i\pm 1}]$ taking value $\pm t^{-2}$ in the bottom left entry and 0 elsewhere.

7.2 Kac-Moody groups

Kac-Moody groups generalise semisimple Lie groups, however, the most general definition of them is highly abstract. We will introduce these objects, relating them to their corresponding Kac-Moody algebras and prove that they admit a twin root datum. Together with results from section 4.5 this will show that Kac-Moody groups act strongly transitively on associated twin buildings. This will provide useful information about the structure of such groups.

Let A be an algebra. A vector space V is called an A-module if there is an algebra homomorphism $\theta : A \to GL(V)$ which commutes with the addition and multiplication in the algebra.

Definition 7.2.1 Types of vector space endomorphism

Let V be a vector space. An endomorphism $T : V \to V$ is said to be

(i) *locally finite if every $v \in V$ lies in some finite-dimensional A-invariant subspace of V,*

(ii) *locally nilpotent if for each $v \in V$ there is some $n \in \mathbb{N}$ such that $T^n(v) = 0$ and*

(iii) *semisimple if V admits a basis of eigenvectors of T.*

Notice that locally nilpotent and semisimple endomorphisms are certainly locally finite.

Let A be a a locally finite endomorphism of V. We can form a corresponding one parameter subgroup of V automorphisms.

$$\exp(tA) = \sum_{n \geq 0} \frac{t^n}{n!} A^n \quad \text{where } t \in \mathbb{C}.$$

Definition 7.2.2 Locally finite elements

Let $\mathfrak{g}$ be a (possibly infinite dimensional) complex Lie algebra and let V be a $\mathfrak{g}$-module which affords the action π.

An element $X \in \mathfrak{g}$ is said to be π-locally finite if $\pi(X)$ is a locally finite endomorphism on V.

The set of all ad*-locally finite elements of $\mathfrak{g}$ is denoted by $F_{\mathfrak{g}}$ and the subalgebra of $\mathfrak{g}$ generated by $F_{\mathfrak{g}}$ is denoted by $\mathfrak{g}_f$.*
$\mathfrak{g}$ is said to be integrable if $\mathfrak{g} = \mathfrak{g}_f$ and a homomorphism $\phi : \mathfrak{h} \to \mathfrak{g}$ between integrable Lie algebras is called integrable if $\phi(F_{\mathfrak{h}}) \subseteq F_{\mathfrak{g}}$.

In a similar way we define $F_{\mathfrak{g},\pi}$ to be the set of all π-locally finite elements of $F_{\mathfrak{g}}$.

A $\mathfrak{g}$-module (V, π) is called integrable if $F_{\mathfrak{g},\pi} = F_{\mathfrak{g}}$. Naturally, the $\mathfrak{g}$-module $(\mathfrak{g}, \mathrm{ad})$ is integrable.

We are now in a position to define the Kac-Moody group associated to $\mathfrak{g}$.

Let G^* be the free group with generating set $F_{\mathfrak{g}}$. For each integrable $\mathfrak{g}$-module $(V, d\pi)$, we define a G^*-module $(V, \widetilde{\pi})$ by

$$\widetilde{\pi}(X) = \exp(d\pi(X)) := \sum_{n \geq 0} \frac{1}{n!}(d\pi(x))^n \quad \text{for } X \in F_{\mathfrak{g}}$$

Set $G = G^*/(\bigcap \ker(\widetilde{\pi}))$ where the intersection is taken over all $d\pi$ corresponding to integrable $\mathfrak{g}$-modules.

The G^*-module $(V, d\pi)$ is naturally a G-module (V, π). We call $(V, d\pi)$ the differential of (V, π) and call G the group associated to $\mathfrak{g}$.

The image of an element $x \in F_{\mathfrak{g}}$ under the canonical homomorphism $G^* \to G$ is denoted by $\exp(x)$. Thus, by definition,

$$\pi(\exp(x)) = \exp(d\pi(x)) := \sum_{n \geq 0} \frac{1}{n!}(d\pi(X))^n$$

for any integrable $\mathfrak{g}$-mofule $(V, d\pi)$. Also notice that $\{\exp tx \,|\, t \in \mathbb{C}\}$ is a 1-parameter subgroup of G.

We define $F_G = \{\exp(x) \,|\, x \in F_{\mathfrak{g}}\}$ and say a G-module $(V, d\pi)$ is differentiable if all elements of F_G act as locally finite endomorphisms on V and $\exp(tx)$ is analytic on any G-invariant finite dimensional subspace of V.

Definition 7.2.3 Kac-Moody groups

Let A be a generalised Cartan matrix and let $\mathfrak{g}'(A)$ be the derived Lie algebra. The Kac-Moody group $G(A)$ associated to A is the group associated to $\mathfrak{g}'(A)$ by the method previously given.

To each real root $a \in \Phi_R$ we define an associated root group $U_a := \exp(\mathfrak{g}_a)$.

In 1983, Dale Peterson and Victor Kac proved that there is a $1-1$ correspondence between integrable $\mathfrak{g}'(A)$ modules and $G(A)$ modules, ([KP83] pages $141 - 166$).

Theorem 7.2.4 Kac-Moody groups admit a twin root datum

We will show that $(G, \{U_a \,|\, a \in \Phi_R\}, H)$ suffices, where $H := \bigcap_{a \in \Phi_R} N_{G(A)}(U_a)$. To ease the notation we write $U^\varepsilon := \langle U_a \,|\, a \in \Phi_R^\varepsilon \rangle$. We follow the proof given in [Ma07].

Proof: The subgroups U_a are nontrivial and generate G so properties (i) and (ii) hold.

Let $b \in \Pi \subseteq \Phi^+$, then U_b can be described as the image of the subgroup of upper unitriangular matrices of $SL_2(\mathbb{C})$ under ψ_i for some i. U^- is generated by the images of lower unitriangular matrices, thus U_b is not contained in U_-, so part (iii) also follows.

Let $\{a, b\}$ be a pair of prenilpotent roots. For ease of notation, define $\mathfrak{g}_{(a,b)} := \langle \mathfrak{g}_c \,|\, c \in (a, b) \rangle$, the vector space generated by these root spaces. We obtain a decomposition,

$$\mathfrak{g}_{(a,b)} = \bigoplus_{c \in (a,b)} \mathfrak{g}_c$$

from the root space decomposition. Define $U_{(a,b)}$ to be the Lie group of $\mathfrak{g}_{(a,b)}$. The multiplication in $U_{(a,b)}$ is given by the Campbell-Baker-Hausdorff formula (1.5.6). Therefore the Lie algebra of the commutator subgroup $[U_a, U_b]$ is contained in $\mathfrak{g}_{a,b}$. Using the exponential mapping we can deduce that the commutator subgroup $[U_a, U_b]$ is contained in the subgroup generated by the U_c with $c \in (a, b)$. Therefore property (v) holds.

Proving part (iv) is more difficult. We use the fact that the span of $\{e_i, f_i, h_i\}$ is a subalgebra $\mathfrak{sl}_i$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. The automorphism $r_i := \exp(\mathrm{ad}(e_i))\exp(\mathrm{ad}(-e_i))\exp(\mathrm{ad}(e_i))$, acts on $\mathfrak{sl}_i$ as the Cartan involution presented in chapter 3. From this we deduce that r_i is a reflection on $\mathfrak{g}(A)$. The remaining details of the proof of part (iv) are more technical and are therefore omitted.

Φ is reduced, so $\Phi \cap (\mathbb{C}a) = \{\pm a\}$ for all $a \in \Phi$, thus (vi) holds vacuously. $\square$

Combining this result with previous ones from section 4.5 we see that $G(A)$ has the Bruhat and Birkhoff decompositions

$$G(A) = \bigsqcup_{w \in W} B^+ w B^+ = \bigsqcup_{w \in W} B^- w B^- = \bigsqcup_{w \in W} B^+ w B^-.$$

Moreover, each Kac-Moody group acts strongly transitively on an associated twin building.

For each real root $a \in \Phi_R$, the Lie algebra $\mathfrak{g}(a)$ generated by $\mathfrak{g}_{\pm a}$ in $\mathfrak{g}'(A)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and contained in $F_{\mathfrak{g}'(A)}$. Therefore the inclusion $\psi_a : \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{g}(a) \hookrightarrow \mathfrak{g}'(A)$ is integrable and thus induces a map $\psi_a : SL_2(\mathbb{C}) \to G(A)$ whose image is denoted by G_a. It can be shown that $G_a = \langle U_a, U_{-a} \rangle$ and that ψ_a is an isomorphism onto its image.

G_a is called the rank one subgroup of the real root a. Similarly, if $a_1, \ldots, a_n \in \Phi_R$ we define $G_{a_1,\ldots,a_n} = \langle G_{a_1}, \ldots, G_{a_n} \rangle$ to be the associated rank n subgroup of these real roots. We now construct a topology on $G(A)$.

Definition 7.2.5 **Kac-Peterson topology**

The final group topology with respect to the inclusion maps $\{\psi_a : SL_2(\mathbb{C}) \to G(A) \,|\, a \in \Phi_R\}$ where $SL_2(\mathbb{C})$ has its connected Lie group topology, is called the Kac-Peterson topology on $G(A)$ and is denoted by $\mathcal{T}_{KP}$.

The aim of the remainder of this section is to prove the following theorem.

Theorem 7.2.6 *The topological group $(G(A), \mathcal{T}_{KP})$ is a k_ω-group.*

We simplify the proof by first giving two lemmas.

Lemma 7.2.7 *Let $\Pi = \{a_1, \ldots a_n\}$ be any choice of simple roots for the Kac-Moody algebra $\mathfrak{g}(A)$, then $\mathcal{T}_{KP}$ is the final group topology with respect to the inclusions $\psi_{a_i} : SL_2(\mathbb{C}) \to G(A)$ for $1 \leq i \leq n$.*

Proof: The Weyl group $W = N_{G(A)}(H)/H$. By construction H normalises every root subgroup so it follows that W acts by conjugation on the root subgroups. In particular, for any root a and any $w \in W$,

$$wU_a w^{-1} = U_{wa} \quad \text{and thus} \quad wG_a w^{-1} = G_{wa}$$

By definition, every real root is of the form wa_i for some $w \in W$ and some $a_i \in \Phi_R$. Conjugation is continuous, so it follows that all rank one subgroup inclusions are continuous. Therefore $\mathcal{T}_{KP}$ is the final group topology with respect to the inclusions $\psi_{a_i} : SL_2(\mathbb{C}) \to G(A)$ for $1 \leq i \leq n$. $\square$

Lemma 7.2.8 *Let $(V, d\psi)$ be an integrable $\mathfrak{g}'(A)$-module, let $\psi : G(A) \to \mathrm{Aut}(V)$ be the associated representation of $G(A)$ and let $a \in \Phi_R$. Every $v \in V$ is contained in a finite dimensional G_a-submodule V_0 of V and the orbit map $G_a \to V$ given by $g \mapsto \psi(g)v$ is continuous onto V_0. Moreover, for each $v \in V$ and real roots $a_1, \ldots, a_n$, the map*

$$G_{a_1} \times \cdots \times G_{a_n} \to V \text{ defined by } (g_1, \ldots, g_n) \mapsto \psi(g_1 g_2 \ldots g_n)v$$

has image contained in some finite dimensional subspace of V and is continuous onto this space.

Proof: V is integrable and $\mathfrak{g}_a \subseteq F_{\mathfrak{g}'(A)}$ which is finite dimensional. [Ka85], proposition 3.6(a) (or [Ka84], lemma (b) on page 170) tells us that each $v \in V$ is contained in a finite dimensional $\mathfrak{g}_a$-submodule V_0 of V. By definition

$$\psi(\exp(X))w = \sum_{k=0}^{\infty} \frac{(d\psi(X))^k}{k!} w \in V_0$$

for all $X \in \mathfrak{g}_a$ and all $w \in V_0$. Thus V_0 is a G_a submodule and $\pi : G_a \to GL(V_0)$ given by $g \mapsto \psi(g)|_{V_0}$ is the smooth representation of G_a with $d\pi(X) = d\psi(X)$ for all $X \in \mathfrak{g}_a$. Therefore the orbit map $G_a \to V_0$, with $g \mapsto \psi(g)v = \pi(g)v$ is continuous.

We now proceed by induction on n, noting that we have completed the case $n = 1$ above. By hypothesis we assume that $\psi(G_{a_2}G_{a_3}\dots G_{a_n})v$ is contained in some finite dimensional vector space $V_0 \subseteq V$ and that the orbit map $\theta : G_{a_2} \times G_{a_3} \times \dots \times G_{a_n} \to V_0$ defined by $\theta(g_2,\dots,g_n) = \psi(g_2,\dots,g_n)v$ is continuous.

Each element $v \in V$ is contained in a finite dimensional $\mathfrak{g}_{a_1}$-module, so the $\mathfrak{g}_{a_1}$-module $V_1 \subseteq V$ generated by V_0 is finite dimensional.

Consider the map $\pi : G_{a_1} \to GL(V_1)$ with $\pi(g) = \theta(g)|_{V_1}$. The preliminary case tells us that π is continuous for each $w \in V_1$ so π is continuous.
The evaluation map $\epsilon : GL(V_1) \times V_1 \to V_1$ which maps (B,x) to $B(x)$ is continuous so it follows that the orbit map

$$\epsilon \circ (\pi \times \psi) : G_{a_1} \times G_{a_2} \times \dots \times G_{a_n} \to V_1 \quad \text{is continuous, as required}$$

Proof of theorem 7.2.6: Fix a system of simple roots $\Pi = \{a_1,\dots,a_n\}$ and for each word $I := (i_1,\dots,i_k)$ over $\{1,\dots,n\}$ define G_I as the image of the product map

$$p_I : G_{a_{i_1}} \times \dots \times G_{a_{i_k}} to G(A) \text{ where } (g_1,\dots,g_k) \mapsto g_1 g_2 \dots g_k.$$

Define the topology $\mathcal{T}_I$ on G_I to be the quotient topology with respect to p_I (the finest topology under which p_I is continuous).

We claim $(G_I, \mathcal{T}_I)$ is a k_ω space. $SL_2(\mathbb{C})$ is a k_ω group so by proposition 6.1.4(iv),

$$SL_2(\mathbb{C})^k \cong G_{a_{i_1}} \times \dots \times G_{a_{i_k}}$$

is a k_ω space, so by part (v) of the same proposition it suffices to show that $(G_I, \mathcal{T}_I)$ is Hausdorff.

To see this let $x, y \in G_I$ be distinct. Then there is some integrable $\mathfrak{g}'(A)$-module $(V, d\psi)$ whose associated representation $\psi : G(A) \to \mathrm{Aut}(V)$ distinguishes x from y, in other words, $\psi(x) \neq \psi(h)$. So for some $v \in V$, $\psi(x)v \neq \psi(y)g$.

Define $f : G_I \to V$ to be such that $f(x) = \psi(x)v$. By lemma 7.2.8, $f \circ P_I$ is continuous, so f is also continuous. Thus we can choose disjoint open neighbourhoods of $f(x)$ and $f(y)$ and their preimages under f will be disjoint open neighbourhoods of x and y in G_I, so G_I is Hausdorff, as required.

Define $\mathcal{W}_n$ to be the poset of all words over the set $\{1,\dots,n\}$ with respect to the ordering $I \le J$ if and only if I is a subsequence of J. For $I \le J$ there is an obvious inclusion map $G_I \hookrightarrow G_J$. Let $\mathcal{T}_0$ be the final topology with respect to the system $(G_I, \mathcal{T}_I)_{I \in \mathcal{W}_n}$ on the union $G(A) = \bigcup G_I$.

Consider the sequence $I_1 \le I_2 \le \dots$ in $\mathcal{W}_n$ defined by $I_1 = (1)$, $I_n = (1,\dots,n)$, $I_{n+1} = (1,\dots,n,1)$ and so on. Then

$$G_{I_1} \subseteq G_{I_2} \subseteq \dots \subseteq G(A)$$

and as each G_{I_n} is k_ω we deduce that $(G(A), \mathcal{T}_0)$ is k_ω. It remains to show that $(G(A), \mathcal{T}_0)$ is a topological group and that $\mathcal{T}_0 = \mathcal{T}_{KP}$.

The concatenation map $G_(i_1, i_2, \dots, i_k) \times G_(j_1, j_2, \dots, j_l) \to G_(i_1,\dots,i_k,j_1,\dots,j_l)$ is continuous, so it is clear that the map $G_(i_1, i_2, \dots, i_k) \times G_(j_1, j_2, \dots, j_l) \to G(A)$ is also continuous. By proposition 6.1.7, group multiplication is continuous. Setting $l = k$ and $i_m = j_{(k+1)-m}$ proves that taking inverses is continuous as well. Thus $(G(A), \mathcal{T}_0)$ is a k_ω group.

Finally, notice that the maps $\psi_i : SL_2(\mathbb{C}) \to G(A)$ are continuous (as they correspond to words of length 1), so by lemma 7.2.7, $\mathcal{T}_0 \subseteq \mathcal{T}_{KP}$. On the other hand, each p_I is continuous with respect to $\mathcal{T}_{KP}$ by lemma 7.2.8. Thus $\mathcal{T}_0 = \mathcal{T}_{KP}$. □

We note one important corollary to this result.

Corollary 7.2.9 *Let $(V, d\psi)$ be an integrable $\mathfrak{g}'(A)$-module and let $\psi : G(A) \to \mathrm{Aut}(V)$ be the corresponding representation on $G(A)$. Then ψ is continuous with respect to the Kac-Peterson topology on $G(A)$ and the topology of pointwise convergence on $\mathrm{Aut}(V)$.*

Proof: (We continue with the notation introduced in the proof of theorem 7.2.6). The restriction of ψ to each G_I is continuous by lemma 7.2.8. Theorem 7.2.6 tells us that $G(A) = \varinjlim(G_I)$ so the result follows. □

7.3 Unitary forms

We will say a subalgebra $\mathfrak{h}$ of $\mathfrak{g}'(A)$ is integrable if and only if the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}'(A)$ is an integrable homomorphism in the sense of definition 7.2.2.

Definition 7.3.1 *A subalgebra $\mathfrak{g}^0(A) \subseteq \mathfrak{g}'(A)$ is called a real form if $\mathfrak{g}'(A) = \mathfrak{g}^0(A) \oplus i\mathfrak{g}^0(A)$.*

A real form $\mathfrak{g}^0(A)$ defines an involution on $\mathfrak{g}'(A)$ given by

$$\sigma(X_0 + iX_1) := X_0 - iX_1 \text{ for } X_0, X_1 \in \mathfrak{g}^0(A)$$

Proposition 7.3.2 *Let $\mathfrak{g}^0(A) \subseteq \mathfrak{g}'(A)$ be a real form of $\mathfrak{g}'(A)$, so $\mathfrak{g}'(A) = \mathfrak{g}^0(A) \oplus i\mathfrak{g}^0(A) \cong \mathfrak{g}^0(A) \otimes \mathbb{C}$. Let $\sigma : \mathfrak{g}'(A) \to \mathfrak{g}'(A)$ be the associated involution. Then $\mathfrak{g}^0(A) \subseteq \mathfrak{g}'(A)$ is an integrable subalgebra, σ is integrable and the induced group involution $\sigma_G : G(A) \to G(A)$ is continuous.*

Proof: Denote by ad and ad_0 the adjoint representations of $\mathfrak{g}'(A)$ and $\mathfrak{g}^0(A)$ respectively. Let $X \in \mathfrak{g}^0(A)$ be ad_0-locally finite and let $Y \in \mathfrak{g}'(A)$. We can write $Y = Y_0 + Y_1$ with $Y_0, Y_1 \in \mathfrak{g}^0(A)$. There exist finite dimensional $\mathrm{ad}_0(X)$-submodules V_0 and V_1 of $\mathfrak{g}^0(A)$ containing Y_0 and Y_1 respectively. Thus $Y \in V_0 \oplus V_1$. Given $Z = Z_0 + iZ_1 \in V_0 \oplus V_1$, we obtain

$$\mathrm{ad}(X)(Z) = [X, Z_0 + iZ_1] = [X, Z_0] + i[X, Z_1] = \mathrm{ad}_0(X)(Z_0) + i\mathrm{ad}_0(X)(Z_1) \in V_0 \oplus iV_1.$$

Thus $V_0 \oplus iV_1$ is a finite dimensional $\mathrm{ad}(X)$ invariant subspace which contains Y. Y was chosen arbitrarily so we deduce that X is ad-locally finite.

Suppose X is ad-locally finite. For $Y \in \mathfrak{g}'(A)$, let V be a finite dimensional $\mathrm{ad}(X)$ invariant subspace of $\mathfrak{g}'(A)$ containing $\sigma(Y)$. Set $V' := \sigma(V)$ and let $Z' \in V'$ with $Z' = \sigma(Z)$ for some $Z \in V$. Then

$$\mathrm{ad}(\sigma(X))(Z') = [\sigma(X), \sigma(Z)] = \sigma([X, Z]) = \sigma(\mathrm{ad}(X)(Z)) \in \sigma(V) = V'.$$

Moreover, $Y = \sigma(\sigma(Y)) \in V'$ and thus V' is a finite dimensional $\mathrm{ad}(\sigma(X))$ invariant subspace containing Y. Again, Y was chosen arbitrarily so we deduce that $\sigma(X)$ is ad-locally finite and thus σ is integrable.

The induced involution $\sigma : G(A) \to G(A)$ is determined by the rule $\sigma(\exp(X)) = \exp(\sigma(X))$. Thus the restriction of σ to any rank one subgroup of $G(A)$ is smooth and hence continuous. Thus σ is continuous, by definition of the Kac-Peterson topology. □

Given a real form $\mathfrak{g}^0(A)$ of $\mathfrak{g}'(A)$, the group $G^0(A) := G(A)^\sigma = \{g \in G(A) \,|\, \sigma(g) = g\}$ is called the real form of $G(A)$ associated with $\mathfrak{g}^0(A)$. The topology on $G^0(A)$ is the subspace topology with respect to $(G(A), \mathcal{T}_{KP})$. We note one immediate corollary of the preceding results.

Corollary 7.3.3 *Every real form of a complex Kac-Moody group is a k_ω group.*

The proof follows from proposition 7.3.2, proposition 6.1.4(iii) and theorem 7.2.6. □

We recall that every complex semisimple Lie algebra $\mathfrak{g}$ has a compact real form $\mathfrak{k}$ which is the fixed point set of the Cartan involution (see proposition 3.4.6). This is used to define the compact real form K of a semisimple Lie group G, from which we obtain the Iwasawa decomposition of G. The Cartan involution restricted to each rank 1 subgroup of G is equivalent to taking the negative conjugate transpose.

A similar construction can be made for Kac-Moody groups. Kac and Peterson, [KP85], proved that an involution can be defined on a derived Lie algebra $\mathfrak{g}'(A)$ which extends the Cartan involution on each rank 1 subalgebra $\mathfrak{g}_a \cong \mathfrak{sl}_2(\mathbb{C})$. This involution is denoted by $\overline{\omega}$ and the subalgebra

$$\mathfrak{k}(A) := \{X \in \mathfrak{g}'(A) \,|\, \overline{\omega}(X) = X\} \subseteq \mathfrak{g}'(A)$$

is called the unitary form of $\mathfrak{g}'(A)$.

The real form $K(A)$ of $G(A)$ corresponding to the unitary form $\mathfrak{k}(A)$ of $\mathfrak{g}'(A)$ is called the unitary form of $G(A)$.

The involution $\overline{\omega} : G(A) \to G(A)$ obtained by integrating $\overline{\omega}$ fixes every rank one subgroup and satisfies

$$K_a := G_a^{\overline{\omega}} = G_a \cap K(A).$$

Moreover, $K(A)$ is generated by the compact Lie groups K_a for $a \in \Phi_R$ and each $K_a \cong SU_2(\mathbb{C})$, the compact real form of $SL_2(\mathbb{C})$.

Corollary 7.3.3 tells us that $K(A)$ is a k_ω group with respect to the subspace topology induced from $(G(A), \mathcal{T}_{KP})$, this topology will also be called the Kac-Peterson topology and denoted by $\mathcal{T}_{KP}$. The following proposition is an analogue of theorem 7.2.6 for unitary forms.

Proposition 7.3.4 *Let $\Pi = \{a_1, \ldots, a_n\}$ be any choice of simple roots for $\mathfrak{g}(A)$. The Kac-Peterson topology $\mathcal{T}_{KP}$ on $K(A)$ is the final group topology with respect to the inclusions $\varphi_{a_i} : SU_2(\mathbb{C}) \hookrightarrow K(A)$ for $i = 1, \ldots, n$.*

Proof: Each K_{a_i} has its compact connected Lie group topology so let $\mathcal{T}_0$ be the final group topology on $K(A)$ with respect to the inclusions $\varphi_{a_i} : K_{a_i} \hookrightarrow K(A)$. We wish to show that $\mathcal{T}_0 = \mathcal{T}_{KP}$.

Each of the maps φ_{a_i} is continuous with respect to $\mathcal{T}_{KP}$ and therefore $\mathcal{T}_{KP} \subseteq \mathcal{T}_0$, so $\mathcal{T}_0$ is Hausdorff.

Using the proof of theorem 7.2.6 as a template we associate a product map φ_I to each word $I = (i_1, \ldots, i_k)$ over $\{1, \ldots, n\}$,

$$\varphi_I : K_{a_1} \times \cdots \times K_{a_k} \to K(A), \qquad (g_1, \ldots, g_k) \mapsto g_1 g_2 \ldots g_k$$

where $a_j := a_{i_j}$ for ease of notation. The image of φ_I is denoted by K_I.

φ_I is continuous with respect to both topologies as both topologies define topological groups on $K(A)$. Moreover, as these topologies are both Hausdorff and $K_{a_1} \times \cdots \times K_{a_k}$ is compact it follows that φ_I is a quotient map for both $\mathcal{T}_0$ and $\mathcal{T}_{KP}$. Thus these topologies coincide on each K_I.

Using the method from the proof of theorem 7.2.6 we can deduce that $\varinjlim(K_I) = (K(A), \mathcal{T}_0)$. Also, $\varinjlim(G_I \cap K(A)) = (K(A), \mathcal{T}_0)$. It is clear that $K_I \subseteq G_I \cap K(A)$ and the topologies coincide on each K_I, so it will be sufficient to prove that for each word $I \in \mathcal{W}_n$ there is some $J \in \mathcal{W}_n$ such that $G_I \cap K(A) \subseteq K_J$. If this is the case then the topologies will coincide on $G_I \cap K(A)$ for all I. Using theorem 6.3 we deduce that $\mathcal{T}_0$ is the subspace topology on $K(A)$ induced from $(G(A), \mathcal{T}_{KP})$, completing the proof.

Denote by s_j the root reflection at a_j. Then $S = \{s_1, \ldots, s_n\}$ generates the Weyl group W. We will say that a word I is special if $(s_{i_1}, \ldots, s_{i_k})$ is a reduced expression for the Weyl group element $w(I) = s_{i_1} \ldots s_{i_k} \in W$.

Suppose I is special. Using the Bruhat decomposition for $SL_2(\mathbb{C})$, we have $G_{a_i} \subseteq B^+ \cup B^+ s_i B^+$. Extending this, we know that

$$G_I \subseteq (B^+ \cup B^+ s_{i_1} B^+) \ldots (B^+ \cup B^+ s_{i_k} B^+) \subseteq \bigcup_{w \leq w(I)} B^+ w B^+.$$

Here $\leq$ describes the Bruhat ordering on W where $w_1 \leq w_2$ if and only if there is a reduced expression of w_2 which contains a reduced expression of w_1 as a subsequence.

By [KP85], proposition 5.1, $K(A) \cap B^+ w B^+ \subseteq K_I.T$ where $T := H \cap K(A)$ is a compact torus. Thus

$$G_I \cap K(A) \subseteq K_I T.$$

Moreover, $T \subseteq K_1 \ldots K_n$ so defining $J = (i_1, \ldots i_k, 1, 2, \ldots, n)$ to be the concatenation of I with $(1, \ldots, n)$, we see that $G_I \cap K(A) \subseteq K_J$. This completes the claim for special $I \in \mathcal{W}_n$.

If $I \in \mathcal{W}_n$ is arbitrary, then define $\mathcal{W}_I$ to be the set of all special subwords of I. $\mathcal{W}_I$ is finite so there is some special word $I' \in \mathcal{W}_n$ such that $W(I') \geq w(J)$ for all $J \in \mathcal{W}_I$. Thus

$$G_I \subseteq \bigcup_{w \leq w(I')} B^+ w B^+.$$

Defining J to be the concatenation of I' with $(1, 2, \ldots, n)$ we see that $G_I \cap K(A) \subseteq K_J$. $\square$

Chapter 8

Phan amalgams

8.1 Borovoi's theorem for complex Kac-Moody groups

Definition 8.1.1 Amalgams

Let $\mathbb{G}$ be the category of groups under group homomorphisms and let $\mathbb{I}$ be the category given by a poset $(J, \leq)$.

A diagram $\delta : \mathbb{I} \to \mathbb{G}$ is called an amalgam if $\delta(a)$ is a group monomorphism for all $a \in \mathrm{mor}(\mathbb{I})$. A cone $(G, (\psi_i)_{i \in \mathrm{ob}(\mathbb{I})})$ over an amalgam δ in $\mathbb{G}$ is called an enveloping group and the corresponding colimit is called a universal enveloping group.

Often an amalgam is written just as a sequence of subgroups when the poset and the monomorphisms are clear from the context.

Borovoi's theorem, 5.3.3, shows that a semisimple Lie group G is the universal enveloping group of its rank 1 and rank 2 subgroups. In this chapter we will prove an analogous result for Kac-Moody groups. To do this we will need to impose some 'finiteness' condition on the Dynkin diagram Γ of a Kac-Moody group $G(A)$. A suitable choice to make is that these diagrams are 2-spherical. This means that the subgraph of Γ induced by any pair of vertices is isomorphic to either $A_1 \oplus A_1$, A_2, B_2 or G_2 (see figure 2.1.2).

Theorem 8.1.2 Borovoi's theorem for complex Kac-Moody groups

Let $K(A)$ be the unitary form of some complex Kac-Moody group $G(A)$ associated with a symmetrisable generalised Cartan matrix of finite size and two-spherical type (W, S). Let Φ_R be the set of real roots and let Π be a system of fundamental roots.

Define a diagram $\delta : \mathbb{I} \to \mathbb{KOG}$ as follows. Choose $\mathbb{I}$ to be the small category with objects $\binom{\Pi}{1} \cup \binom{\Pi}{2}$ and morphisms $\{a\} \to \{a, b\}$ for all $a, b \in \Pi$ with $a \neq b$. Define $\delta(\{a\}) = K_a$, $\delta(\{a, b\}) = K_{ab}$ and $\delta(\{a\} \to \{a, b\}) = (K_a \hookrightarrow K_{ab})$.

Then $((K(A), \mathcal{T}_{KP}), (\iota_i)_{i \in \binom{\Pi}{1} \cup \binom{\Pi}{2}})$ (where ι_i is the identity inclusion map) is a colimit of the diagram δ in the categories of abstract groups, Hausdorff topological groups and k_ω groups.

Proof: The proof that δ is a colimit in the category of abstract groups is exactly the same as the proof of theorem 5.3.3, using the fact that $G(A)$ acts strongly transitively on an associated twin building and we can define a fundamental chamber with respect to the action of $K(A)$. Applying extensions of Tits' lemma and the Solomon-Tits theorem to twin buildings will complete the proof. $(K(A), \mathcal{T}_{KP})$ is Hausdorff by proposition 7.3.4 and is a k_ω group by proposition 7.3.4 and corollary 6.2.6. $\square$

Amalgams of the type considered by theorem 8.1.2 are called standard amalgams. Our final major result will be to prove that under certain circumstances this amalgam is unique up to isomorphism.

8.2 Standard Phan amalgams

Definition 8.2.1 Topological weak Phan systems

Let Γ be a two-spherical Dynkin diagram with vertex set Π and let G be a topological group. A family $(\overline{K_j})_{j\in\Pi}$ of subgroups of G is called a topological weak Phan system of type Γ over $\mathbb{C}$ in G if the following properties hold.

(i) *G is generated by $\bigcup_{j\in\Pi}\overline{K_j}$,*

(ii) *each $\overline{K_j}$ is isomorphic (as a topological group) to a central quotient of a simply connected, compact, semisimple Lie group of rank one,*

(iii) *given any two distinct elements $i, j \in \Pi$, the subgroup $\overline{K_{ij}} := \langle \overline{K_i}, \overline{K_j}\rangle$ of G is isomorphic as a topological group to a central quotient of the simply connected, compact, semisimple Lie group of rank two corresponding to the rank two subdiagram of Γ induced by the vertices i and j,*

(iv) *$(\overline{K_i}, \overline{K_j})$ or $(\overline{K_j}, \overline{K_i})$ is a standard pair in $\overline{K_{ij}}$.*

This definition requires Γ to be two-spherical, otherwise condition (iii) can never hold.

Definition 8.2.2 Phan amalgams

Let Γ be a two-spherical Dynkin diagram with vertex set Π. An amalgam $\mathcal{A} = (\overline{K_j})$ for $j \in \binom{\Pi}{1} \cup \binom{\Pi}{2}$ is called a Phan amalgam of type Γ over $\mathbb{C}$, if for all distinct pairs $a, b \in \Pi$, the group $K_{a,b}$ is a central quotient of a simply connected compact semisimple Lie group whose type is given by the subgraph of Γ induces by the vertices a and b and its morphisms are the maps $\overline{K_a} \hookrightarrow \overline{K_{ab}}$ which embed $(\overline{K_a}, \overline{K_b})$ as a standard pair into K_{ab}.

It is clear from the definition that every topological weak Phan system gives rise to a Phan amalgam. The converse is not true, as the definition of a Phan amalgam makes no mention of the amalgam possessing an enveloping group. However, if such a group G exists into which $\mathcal{A}$ can be embedded, then the image of $\mathcal{A}$ in G is a weak Phan system of G. To distinguish this case, we call such Phan amalgams strongly noncollapsing.

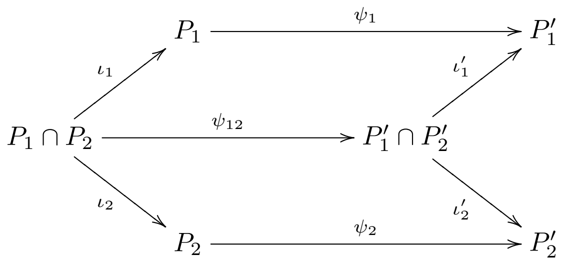
Definition 8.2.3 Irreducible Phan amalgams

A Phan amalgam $\mathcal{A}$ is said to be irreducible if and only if its Dynkin diagram is connected.

Definition 8.2.4 Amalgam isomorphisms

Let $\mathcal{A} = (P_1 \overset{\iota_1}{\hookleftarrow} (P_1 \cap P_2) \overset{\iota_2}{\hookrightarrow} P_2)$ and $\mathcal{A}' = (P_1' \overset{\iota_1'}{\hookleftarrow} (P_1' \cap P_2') \overset{\iota_2'}{\hookrightarrow} P_2')$ be two amalgams of abstract groups. The amalgams are said to be of the same type if there exist isomorphisms $\psi_i : P_i \to P_i'$ with the property that $(\psi_i \circ \iota_i)(P_1 \cap P_2) = \iota_i'(P_1' \cap P_2')$ for $i = 1, 2$.

If there is also an isomorphism $\psi_{12} : P_1 \cap P_2 \to P_1' \cap P_2'$ such that the following diagram commutes,



then the amalgams $\mathcal{A}$ and $\mathcal{A}'$ are said to be isomorphic. We denote an amalgam isomorphism as a triple $(\psi_1, \psi_{12}, \psi_2)$.

Here each map ι denotes the identity map and they differ only in their domain and codomain.

If the amalgams consist of topological groups, they are said to be isomorphic if and only if there is an amalgam isomorphism $(\psi_1, \psi_{12}, \psi_2)$ for the underlying amalgam of abstract groups in which each of the three maps is also a homeomorphism.

Lemma 8.2.5 Goldschmidt's lemma

Let $\mathcal{A} = (P_1 \stackrel{\iota_1}{\hookleftarrow} (P_1 \cap P_2) \hookrightarrow^{\iota_2} P_2)$ *be an amalgam of topological groups. For* $i = 1, 2$ *define* $A_i = \mathrm{Stab}_{\mathrm{Aut}(P_i)}(P_1 \cap P_2)$ *and let* $a_i : A_i \to \mathrm{Aut}(P_1 \cap P_2)$ *map* $f \in A_i$ *onto its restriction to* $P_1 \cap P_2$*.*

There is a one-to-one correspondence between isomorphism classes of amalgams of the same type as $\mathcal{A}$ *and the double cosets*

$$a_2(A_2)\backslash\mathrm{Aut}(P_1 \cap P_2)/\, a_1(A_1).$$

Proof: By definition 8.2.4 any amalgam of the same type as $\mathcal{A}$ is isomorphic to an amalgam of the form

$$(P_1 \stackrel{\iota_1 \circ b_1}{\hookleftarrow} (P_1 \cap P_2) \stackrel{\iota_2 \circ b_2}{\hookrightarrow} P_2)$$

where $b_1, b_2 \in \mathrm{Aut}(P_1 \cap P_2)$. This in turn is isomorphic to the amalgam

$$(P_1 \stackrel{\iota_1}{\hookleftarrow} (P_1 \cap P_2) \stackrel{\iota_2 \circ b_2 \circ b_1^{-1}}{\hookrightarrow} P_2)$$

via the amalgam isomorphism $(\mathrm{id}, b_1, \mathrm{id})$.

Thus it remains to determine when the following two amalgams are isomorphic.

$$(P_1 \stackrel{\iota_1}{\hookleftarrow} (P_1 \cap P_2) \stackrel{\iota_2 \circ c_1}{\hookrightarrow} P_2)$$

$$(P_1 \stackrel{\iota_1}{\hookleftarrow} (P_1 \cap P_2) \stackrel{\iota_2 \circ c_2}{\hookrightarrow} P_2)$$

To achieve this we need an amalgam isomorphism $(\psi_1, \psi_{12}, \psi_2)$ with the properties $\iota_1 \circ \psi_{12} = \psi_1 \circ \iota_1$ and $\iota_2 \circ c_2 \circ \psi_{12} = \psi_2 \circ \iota_2 \circ c_1$. From the first equality it is clear that $\psi_{12} \in a_1(A_1)$ and the second says $c_2 \circ \psi_{12} \circ c_1^{-1} \in a_2(A_2)$. Therefore such an amalgam exists if and only if $a_1(A_1) \cap c_2^{-1} a_2(A_2) c_1 \neq \emptyset$. This is equivalent to requiring

$$a_2(A_2)c_1 a_1(A_1) = a_2(A_2)c_2 a_1(A_1).$$

Definition 8.2.6 Umambiguous Phan amalgams

A Phan amalgam $(\overline{K_{ab}})$ *and any subamalgam of a Phan amalgam is said to be unambiguous if every* $\overline{K_{ab}}$ *is isomorphic to the corresponding rank two semisimple Lie group* K_{ab}*.*

8.3 Classification

Theorem 8.3.1 Classification

Let $n \geq 2$ *and let* $\mathcal{A}$ *be a tree-like strongly noncollapsing unambiguous irreducible Phan amalgam of rank* n*. Then* $\mathcal{A}$ *is unique up to isomorphism.*

We have already seen from theorem 8.1.2 that the rank one and two subgroups of the unitary form $K(A)$ of a Kac-Moody group whose Dynkin diagram is a 2-spherical tree forms such an amalgam. We therefore deduce from this theorem that there amalgams are unique up to isomorphism.

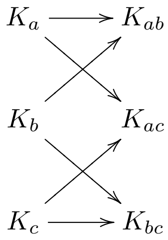
To consider an amalgam as strongly noncollapsing we recall that its Dynkin diagram must be two-spherical. Moreover, the condition that $\mathcal{A}$ is tree-like and irreducible says that the Dynkin diagram is in fact a tree. Critically, this guarantees that given any collection of at least three vertices from the corresponding Dynkin diagram Γ the subdiagram induced by some pair $\{a, b\}$ of them has type $A_1 \oplus A_1$, so the group K_{ab} is isomorphic to the direct product $K_a \times K_b$.

Proof: Consider a tree-like strongly noncollapsing unambiguous irreducible Phan amalgam

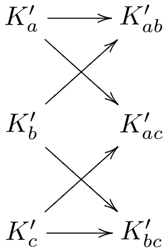
$$\mathcal{A} = (K_J)_{J \in \binom{\Pi}{1} \cup \binom{\Pi}{2}}$$

of rank $n \geq 2$. The proof proceeds by induction on n. As $\mathcal{A}$ is 2-spherical the amalgams of rank two are unique by theorem 2.1.1.

For $n = 3$, let $\Pi = \{a, b, c\}$ and let $\mathcal{A}$ and $\mathcal{A}'$ be the amalgams



and



respectively.

As we have assumed $\mathcal{A}$ and $\mathcal{A}'$ are both irreducible, without loss we may also assume that they both have the Dynkin diagram

$$a \text{ —— } b \text{ —— } c$$

and the two edges have labels contained in the set $\{3, 4, 6\}$. This condition on the labels holds as the Dynkin diagram is 2-spherical and connected.

By Goldschmidt's lemma 8.2.5, the amalgams

$$\mathcal{B} = (K_{ab} \hookleftarrow K_b = K_{ab} \cap K_{bc} \hookrightarrow K_{bc}) \quad \text{and} \mathcal{B}' = (K'_{ab} \hookleftarrow K'_b = K'_{ab} \cap K'_{bc} \hookrightarrow K'_{bc})$$

are isomorphic via some amalgam isomorphism ψ. We do not need to write three isomorphisms as every isomorphism of K_b is induced by some automorphism of K_{ab}, ([HM98] theorem 6.73 gives the details). Therefore we can deduce that $\psi(K_b) = K'_b$.

The groups K_a and K_b are clearly a standard pair in K_{ab}, and hence $(\psi(K_a), \psi(K_b))$ forms a standard pair in $K'_{ab} = \psi(K_{ab})$. Standard pairs are conjugate, so there exists an automorphism of K'_{ab} which maps $\psi(K_a)$ onto K'_a and leaves $K'_b = \psi(K_b)$ invariant. Relabelling ψ as the composition of this automorphism with the original ψ, we may assume that $\psi(K_a) = K'_a$.

Define $D^i_j := N_{K_j}(K_i)$, where K_i and K_j are considered as subsets of K_{ij}. As (K_i, K_j) is a standard pair in K_{ij}, D^i_j is a maximal torus in K_j. If $i = a, c$, then $D^i_b = C_{K_b}(D^b_i)$.

As $\mathcal{A}$ is strongly noncollapsing, let $\pi : \mathcal{A} \to G$ define an enveloping group of G where π is injective on each K_i. To consider π acting on the K_i we have implicitly used the assumption that $\mathcal{A}$ is unambiguous.

π acts as a group monomorphism on each K_i so it follows that $\pi(D^i_b) = C_{\pi(K_b)}(\pi(D^b_i))$. Moreover, $\pi(K_b)$ and $\pi(D_c)$ are invariant under $\pi(D^a_b) = N_{\pi(K_a)}(\pi(K_b))$, so $\pi(D^c_b)$ is invariant under $\pi(D^a_b)$.

Therefore, we have deduced that a maximal torus $\pi(D^a_b)$ of $\pi(K_a)$ leaves the maximal tori $\pi(D^a_b)$ and $\pi(D^c_b)$ of $\pi(K_b)$ invariant.

In this situation, analysis of the rank 2 subgroup K_{ab} yields the result $D^a_b = D^c_b$, which we will now denote by D_b.

$N_{K_b}(K_a) = D_b$, $\psi(D_b) = D'_b := N_{K'_b}(K'_a) = N_{K'_b}(K'_c)$, since $\psi(K_a) = K'_a$ and $\psi(K_b) = K'_b$.

Returning to the Dynkin diagram we have identified a root system of type A_2, B_2 or G_2 with respect to which the groups $K'_b = \psi(K_b)$, K'_c and $\psi(K'_c)$ all occur as compact rank one subgroups. Using [Ca05], plate X, we can deduce that there exists an automorphism of K'_{bc} which centralises K'_b and maps K'_c onto $\psi(K_c)$, again we may compose this automorphism with ψ and assume that $\psi(K_c) = K'_c$. Finally, $\psi(K_{ac}) = K'_{ac}$ as $K'_{ac} \cong K_a \times K_c$ and $K'_{ac} \cong K'_a \times K'_c$.

Let $|\Pi| \geq 4$ and assume as our inductive hypothesis that the result holds for all amalgams of smaller rank.
Let $\mathcal{A} = (K_J)_{J \in \binom{\Pi}{1} \cup \binom{\Pi}{2}}$ be a tree-like strongly noncollapsing unambiguous irreducible Phan amalgam of rank $|\Pi|$. Now let $a, b \in \Pi$ be two nonadjacent vertices of the corresponding Dynkin diagram Γ with the property Γ remains connected (and hence a tree) if these two vertices are removed. This is always possible as the amalgam is tree-like and irreducible, with $|\Pi| \geq 3$.

The amalgams $\mathcal{A}_i = (K_J)_{J \in \binom{\Pi \setminus \{i\}}{1} \cup \binom{\Pi \setminus \{i\}}{2}}$ for $i = a, b$, are isomorphic to standard Phan amalgams by the inductive hypothesis. Therefore the colimits, H_a of $\mathcal{A}_a$ and H_b of $\mathcal{A}_b$ are both determined by theorem 8.1.2.

Now suppose $\mathcal{A} = (K_J)_{J \in \binom{\Pi}{1} \cup \binom{\Pi}{2}}$ and $\mathcal{A}' = (K'_J)_{J \in \binom{\Pi}{1} \cup \binom{\Pi}{2}}$ are two tree-like strongly noncollapsing unambiguous irreducible Phan amalgam of rank $|\Pi|$ with the same Dynkin diagram Γ. Choose two nonadjacent vertices a, b of this Dynkin diagram such that Γ remains connected when these two vertices are removed. Maintaining the notation, we notice that $H_a \cong H'_a$ and $H_b \cong H'_b$ by assumption.

Set $\mathcal{B} = (K_J)_{J \in \binom{\Pi \setminus \{a,b\}}{1} \cup \binom{\Pi \setminus \{a,b\}}{2}}$ and $\mathcal{B}' = (K'_J)_{J \in \binom{\Pi \setminus \{a,b\}}{1} \cup \binom{\Pi \setminus \{a,b\}}{2}}$,

then again by the inductive hypothesis $\mathcal{B} \cong \mathcal{B}'$ and the colimits H_0 of $\mathcal{B}$ and H'_0 of $\mathcal{B}'$ are determined by theorem 8.1.2.

Then the amalgams $H_a \hookleftarrow H_0 \hookrightarrow H_b$ and $H'_a \hookleftarrow H'_0 \hookrightarrow H'_b$ have the same type. Applying [Ca05] theorem 8.2 and Goldschmidt's lemma, (8.2.5), the amalgams $H_a \hookleftarrow H_0 \hookrightarrow H_b$ and $H'_a \hookleftarrow H'_0 \hookrightarrow H'_b$ are in fact isomorphic under some map ψ. The standard Phan amalgams of $\psi(\mathcal{B})$ and $\mathcal{B}'$ in H'_0 correspond to two choices of a maximal torus in H'_0, so these maximal tori are conjugate by the Iwasawa decomposition. Therefore there is some inner automorphism of H'_0 which maps $\psi(K_i)$ to K'_i for all $i \in \binom{\Pi \setminus \{a,b\}}{1} \cup \binom{\Pi \setminus \{a,b\}}{2}$.

Studying H'_a and H'_b we see that $\psi(K_a) = K'_a$ and $\psi(K_b) = K'_b$. Also $\psi(K_{ij}) = K'_{ij}$ for $i = a, b$ and $j \in \Pi \setminus \{a, b\}$. By assumption $K_{ab} \cong K_a \times K_b$, so ψ induces an isomorphism between $\mathcal{A}$ and $\mathcal{A}'$, completing the proof. □

Chapter 9

Conclusion

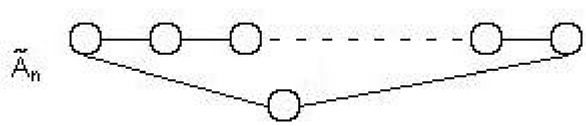
We have seen that every compact simply connected semisimple Lie group K is the universal enveloping group of the amalgam of rank one and rank two subgroups. Moreover, the Lie group topology on K is determined by the induced topology on these rank one and rank two subgroups.

Following this, we extended the result to unitary forms of Kac-Moody groups over symmetrisable generalised Cartan matrices, again deducing that they are the universal enveloping group of the Phan amalgam consisting of their rank one and rank two subgroups.

Finally, we proved that under suitable conditions, which force the Dynkin diagram of a Phan amalgam to be a tree, the unitary forms of Kac-Moody groups are in fact the universal enveloping groups of the unique Phan amalgam (up to isomorphism) with that particular Dynkin diagram.

More recent research concentrates on what conclusions can be made about both Phan amalgams and related Curtis-Tits amalgams under the hypothesis of theorem 8.3.1 when the tree-like condition is removed. In particular, Phan amalgams of type $\widetilde{A_n}$ are of interest, as they form one of the easiest examples of so-called circular amalgams. We have seen that the rank one and rank two subgroups of $\mathfrak{sl}_{n+1}(\mathbb{C}[t,t^{-1}])$ define a Phan amalgam which has Dynkin diagram $\widetilde{A_n}$, but this amalgam is certainly not unique up to isomorphism.

Recalling the diagram $\widetilde{A_n}$,



we can apply the classification theorem 8.3.1 to any copy of A_n contained inside it, so the amalgam is unique up to isomorphism on this path, but then we are left with the question of how the two 'ends' are joined.

In his 1999 paper, [Mü99], Bernhard Mühlherr considers 2-spherical circular Dynkin diagrams, where the edges can be labelled with one of the values 3, 4 or 6. His method is to consider this cycle as an infinite path and consider the ways in which it can be folded up.

$$\cdots \text{———} a_1 \text{———} \cdots \text{———} a_n \text{———} a_1 \text{———} \cdots \text{———} a_n \text{———} \cdots$$

where each of the vertices a_1 up to a_n constitutes a copy of $SU_2(\mathbb{C})$ living inside the group.

One of the outcomes of this paper is that the amalgam is then unique up to a field automorphism, so a maximal set of non-isomorphic amalgams of this type is in bijective correspondence with $\mathrm{Aut}(\mathbb{C})$.

Using this paper and the methods and results it contained, Riuwert Blok and Corneliu Hoffman, ([BH09a] and [BH09b]) extend this idea to the case when a group can be considered as acting strongly transitively on an associated twin building yet also be able to interchange the two halves of that twin building. It is immediate in this case that the field automorphism should still play a role and the question remains as to how much extra freedom can be achieved.

Again we may use the classification theorem (8.3.1) on any subdiagram which is a tree. Considering the ways of joining the last two edges they proved that the field automorphisms remain and also the last edge can act as a transpose inverse (like the Cartan involution on $SU_2(\mathbb{C})$), which swaps the two halves of the building.

This transpose inverse flips the defined notion of positivity, meaning that a positive root group (the upper unitriangular matrices in some copy of $SU_2(\mathbb{C})$) gets mapped to positive root groups as it moves along the path A_n but when it arrives at the last vertex, the transpose inverse maps it to a negative root group (the lower unitriangular matrices in some copy of $SU_2(\mathbb{C})$). Therefore, the corresponding fundamental chamber sees both halves of the building, leading to a natural comparison with a Möbius band.

The result of this is that there is a bijective correspondence between non-isomorphic noncollapsing amalgams of this type, (noncollapsing meaning that they do in fact define some nontrivial group) and the group $\mathrm{Aut}(\mathbb{C}) \times \mathbb{Z}_2$.

The first major outstanding question which has set to be resolved is what conclusion can be reached in theorem 8.3.1 if the tree-like condition is removed. Recent results give answers in the case of circular Phan amalgams, but there is no definitive answer for some arbitrary generalised Cartan matrix. This would help to cover more cases of standard Phan amalgams, describing the amount of freedom such amalgams can take. It is known in the case of a cycle that this is given by the automorphism group of the underlying field, so the question is what happens when two cycles meet. Suitable conditions are needed to ensure that the field automorphisms agree, so the expected result should be a bijective correspondence between non-isomorphic amalgams of the same type and some carefully chosen subgroup of $\mathrm{Aut}(\mathbb{C})$, which is trivial for connected tree-like Dynkin diagrams and the entire group for cycles.

Another extension that has been made to much of the theory covered by this project is to consider the same constructions over finite fields, replacing Lie groups with finite groups of Lie type. Since the classification of finite simple groups it has been known that simple groups of Lie type form the least well understood of the three infinite families of finite simple groups.

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