# Kazhdan's Property (T)

## Talk 1: Definitions, examples and non-examples.

David Hume, University of Bristol
david.hume@bristol.ac.uk

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# Disclaimer

The book "Kazhdan's Property (T)" by Bekka, de la Harpe and Valette is an excellent (and freely available) resource.

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My goal is to demystify some of the definitions, explain strategies to prove that a group does or doesn’t have property (T), and survey some of the many applications of property (T).

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2) If a locally compact group $G$ has property (T), then all lattices in $G$ have property (T).
3) A discrete countable group which has property (T) admits a finite generating set and has finite abelianisation.

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To find such a property, Kazhdan considered unitary representations of groups. Why?

# Representations

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A unitary representation of $G$ in $\mathcal{H}$ is a group homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$ which is strongly continuous, meaning that for each $x \in \mathcal{H}$, the map

$$G \to \mathcal{H} \quad \text{given by} \quad g \mapsto \pi(g)x$$

is continuous.

# Examples of Unitary Representations

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Given a locally compact group $G$ with left invariant Haar measure $dg$, the left regular representation $\lambda_G : G \to \mathcal{U}(L^2(G, dg))$ given by $\lambda_G(g)F(x) = F(g^{-1}x)$ for all $F \in L^2(G, dg)$ and $x \in G$ is a unitary representation.

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We have $\mathbf{1} \subset \lambda_G$ if and only if $G$ is compact.

# Almost invariant vectors

Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation, let $K$ be a subset of $G$ and let $\varepsilon > 0$. A vector $v \in \mathcal{H}$ is $(K, \varepsilon)$-invariant if

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$\pi$ has almost invariant vectors if, for every compact $K \subset G$ and $\varepsilon > 0$ there is a non-zero $(K, \varepsilon)$-invariant vector $v_{K,\varepsilon}$.

# Examples

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**Proof.**

Fix some compact $K \subset \mathbb{R}$ and some $\varepsilon > 0$. Choose $a \in [0, \infty)$ such that $K \subset [-a\varepsilon^2, a\varepsilon^2]$ and define $v = \chi_{[-4a,4a]}$.
For each $g \in K$, $|\pi(g)v - v\| \leq \chi_B$, where $B = [-a(4+\varepsilon^2), -a(4-\varepsilon^2)] \cup [a(4-\varepsilon^2), a(4+\varepsilon^2)]$. Thus

$$\sup_{g \in K} \|\pi(g)v - v\| \leq \|\chi_B\| = (4a\varepsilon^2)^{\frac{1}{2}} < \varepsilon(8a)^{\frac{1}{2}} = \varepsilon\|v\|.$$

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## Lemma

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$$\sup_{g \in K} \|\pi(g)v - v\| \leq \|\chi_B\| = (4a\varepsilon^2)^{\frac{1}{2}} < \varepsilon(8a)^{\frac{1}{2}} = \varepsilon\|v\|. \qquad \square$$

More generally, a group $G$ is amenable if and only if the left regular representation has almost invariant vectors.

# Property (T)

A locally compact group $G$ has Property (T) if there exists a compact subset $K \subset G$ and some $\varepsilon > 0$ such that every unitary representation with $(K, \varepsilon)$-almost invariant vectors contains the trivial representation (has a non-zero fixed vector).

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**Theorem**

*A locally compact amenable group with property $(T)$ is compact.*

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**Theorem**

*A locally compact amenable group with property (T) is compact.*

**Proof.**

We previously established that $G$ is amenable if and only if the left regular representation has almost invariant vectors, and the left regular representation of $G$ contains the trivial representation if and only if $G$ is compact. □

# Consequences of Property (T): Quotients

**Theorem**

*Let $\psi : G \to H$ be a continuous homomorphism between topological groups with dense image. If $G$ has property (T) then $H$ has property (T).*

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**Proof.**

Let $Q \subset G$ be compact and $\varepsilon > 0$ be given by the definition of property (T) for $G$. Set $Q' = \psi(G)$ which is a compact subset of $H$.

Let $\pi : H \to \mathcal{U}(\mathcal{H})$ be a unitary representation with a $(Q', \varepsilon)$ invariant vector $v'$. Now $\pi \circ \psi : G \to \mathcal{U}(\mathcal{H})$ is a unitary representation of $G$ and $v'$ is $(Q, \varepsilon)$ invariant. Hence there is some non-zero vector $v$ which is invariant under $\pi(\psi(G))$. Since $\psi(G)$ is dense in $H$ and $\pi$ is strongly continuous, $v$ is invariant under $H$. □

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**Proof.**

Let $\mathcal{C}$ be the set of all open compactly generated subgroups of $G$. For each $H \in \mathcal{C}$, we have that $G/H$ is discrete, so there is a unitary representation

$$\lambda_{G/H} : G \to \ell^2(G/H) \quad \text{given by} \quad \lambda_{G/H}(x)F(gH) = F(x^{-1}gH).$$

Now define $\pi = \bigoplus_{H \in \mathcal{C}} \lambda_{G/H}$. Suppose $\pi$ has a non-zero invariant vector $v = \bigoplus_{H \in \mathcal{C}} v_H$. Choose $H$ so that $v_H \neq 0$. Now $v_H$ is a non-zero $G$-invariant vector in $\ell^2(G/H)$ so must be constant. Hence $G/H$ is finite, and $G$ is compactly generated. □

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**Claim**

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**Proof.**

We prove that it has almost invariant vectors. As $G$ has property (T) we deduce that there is an invariant vector.
Fix $K \subset G$ compact. Since $\mathcal{C}$ is an open cover of $G$, we have $K \subset H_1 \cup \ldots \cup H_n$ for some $H_i \in \mathcal{C}$. Now $H = \langle H_1, \ldots, H_n \rangle \in \mathcal{C}$.
Define $\delta_H \in \ell^2(G/H)$ to be the Dirac function. We have

$$\|\pi(x)\delta_H - \delta_H\| = 0 \quad \text{for all} \quad x \in K$$

viewing $\delta_H$ as a vector in $\bigoplus_{H \in \mathcal{C}} \ell^2(G/H)$. □

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*Proof.*

The map $G \to G^{ab}$ is continuous and surjective. Thus $G^{ab}$ has property (T). It is also abelian, and hence amenable. Therefore, $G^{ab}$ is compact. □

As a corollary, non-abelian free groups do not have property (T).

# Groups with Property (T): Compact Groups

**Theorem**

*Let $G$ be a topological group. If a unitary representation of $G$ admits a $(G, \sqrt{2})$-invariant vector, then it has an invariant vector. In particular, every compact group $G$ has property $(T)$.*

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**Proof.**

Let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a unitary representation admitting a $(G, \sqrt{2})$-invariant vector $v$. Let $\mathcal{C}$ be the closed convex hull of $\pi(G)v$. Let $v_0$ be the unique point in $\mathcal{C}$ with minimal norm.

Since $\pi(g)\mathcal{C} = \mathcal{C}$ for all $g \in G$, we have $\pi(g)v_0 = v_0$ for all $g \in G$.

Set $\varepsilon = \sqrt{2} - \sup_{g \in G} \|\pi(g)v - v\| > 0$. By the parallelogram law

$$2 - 2Re\langle \pi(g)v, v\rangle = \|\pi(g)v - v\|^2 \leq (\sqrt{2} - \varepsilon)^2$$

for all $g \in G$. Hence $Re\langle \pi(g)w, w\rangle \geq \frac{\varepsilon}{2}(2\sqrt{2} - \varepsilon) > 0$ for all $w \in \mathcal{C}$. Thus $v_0 \neq 0_{\mathcal{H}}$. □

Starting point: if a unitary representation of $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ has almost invariant vectors, then there is a vector which is invariant under the action of $\mathbb{Z}^2$ (we say the pair $(SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ satisfies relative Property (T)).

# Groups with Property (T): Bounded Generation (Shalom)

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A group $G$ has bounded generation if there is a finite subset $S \subset G$ and a positive integer $N$ such that for every $g \in G$ there is an equality $g = \prod_{i=1}^{N} s_i^{r_i}$ where each $s_i \in S$ and $r_i \in \mathbb{Z}$.

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Step 1: Let $\pi : SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \to \mathcal{U}(\mathcal{H})$ be a unitary representation. If $v$ is $(Q, \epsilon/20)$-invariant, then it is $(\mathbb{Z}^2, \epsilon)$-invariant.

Step 2: If $G$ is boundedly generated by a set $S$ which can be "covered" by copies of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$, then $G$ has Property (T).

# Groups with Property (T): Group Rings (Ozawa)

Given a group $G$, generated by a finite symmetric set $S$, the group ring $\mathbb{R}G$ is the vector space of finitely support functions $G \to \mathbb{R}$.

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The Laplacian $\Delta \in \mathbb{R}G$ is defined by

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A group $G$ satisfies Kazhdan's Property (T) if there exists some $\lambda > 0$ and finitely many elements $v_i \in \mathbb{R}G$ such that

$$\Delta^2 - \lambda\Delta = \sum_i v_i^* v_i.$$

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