Kazhdan's Property (T)

Talk 2: More examples and applications.

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Recap: Definitions

Let G be a topological group and let $\mathcal{H}$ be a complex Hilbert space. $\mathcal{U}(\mathcal{H})$ is the unitary group of bounded invertible linear operators which preserve the inner product.

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A locally compact group G has Property (T) if there exists a compact subset $K \subset G$ and some $\varepsilon > 0$ such that every unitary representation with (K,ε)-almost invariant vectors has a non-zero fixed vector.

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A locally compact G with property (T) is compactly generated, and has compact abelianisation. If G is also amenable then it is compact.

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The following groups have Property (T):

- Simple Lie groups of real rank at least 2 and their lattices.
- $Aut(F_n)$, $n \geq 5$ (and possibly also $n = 4$).

Actions on Hilbert spaces and cocycles

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$b : G \to \mathcal{H}$ satisfies the 1-cocycle relation $b(gh) = b(g) + \pi(g)b(h)$.

Cocycles and coboundaries

Define $Z^1(G, \pi)$ to be the vector space of continuous maps $G \to \mathcal{H}$ satisfying the 1-cocycle relation $b(gh) = b(g) + \pi(g)b(h)$. Elements of $Z^1(G, \pi)$ are called 1-cocycles with respect to π.

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A 1-cocycle b is called a 1-coboundary with respect to π if there exists a $v \in \mathcal{H}$ such that

$$b(g) = v - \pi(g)v \quad \text{for all} \quad g \in G.$$

$B^1(G,\pi)$ is the vector space of 1-coboundaries.

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Usual cohomology (with real coefficients) corresponds to the case where π is the trivial action on the Hilbert space $\mathbb{R}$.

Lemma

If $\alpha : G \to Isom(\mathcal{H})$ has a bounded orbit, then it has a fixed point.

Property (FH) and bounded cohomology

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If $\alpha : G \to Isom(\mathcal{H})$ has a bounded orbit, then it has a fixed point.

The first bounded cohomology group with coefficients in π is

$$\overline{H^1(G,\pi)} = Z^1(G,\pi)/\overline{B^1(G,\pi)}.$$

Property (FH) and Property (T)

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Every countable group is a subgroup of a finitely generated group which satisfies Property (T).

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Extremely local criterion for Property (T) [Żuk]

Let G be a group with a finite symmetric generating set S which does not contain e. The graph $L(S)$ is the link of e in the Cayley graph $\Gamma(G,S)$, i.e. it has vertex set S and edges ss' whenever $s, s', s^{-1}s' \in S$. We assume $L(S)$ is connected.

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The Laplace operator $\Delta : \ell^2(S) \to \ell^2(S)$ is given by

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Any (non-zero) constant function is an eigenfunction with eigenvalue 0. Let λ_1 be the smallest positive eigenvalue of Δ.

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λ_1 and the Cheeger constant are related by the following:

Theorem

$$2h(\Gamma) \geq \lambda_1(\Gamma) \geq \frac{h(\Gamma)^2}{2\Delta(\Gamma)}.$$

In particular, $\lambda_1(\Gamma) > \frac{1}{2}$ *whenever* $h(\Gamma) \geq \sqrt{\Delta(\Gamma)}$.

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Expanders

Let G be a group with Property (T) which is generated by a finite symmetric set S. Let $\phi : G \to Q$ be a surjective homomorphism with Q finite. Let $A \subset Q$. There is some unit vector $v \in \ell^2(Q)$ such that

$$\frac{|\partial A|}{\min\{|A|, |A^c|\}} \geq \frac{1}{4} \max_{s \in S} \|\pi_Q(s)v - v\|^2.$$

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