Quantum Computing Applications

Ashley Montanaro

Department of Computer Science,
University of Bristol

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University of BRISTOL

Introduction

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2. More recent applications
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The Quantum Algorithm Zoo (http://math.nist.gov/quantum/zoo/) cites 209 papers on quantum algorithms alone, so this is necessarily a partial view…

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- and algorithms which run in time exponential in the input size (i.e. time $O(2^{n^\delta})$ for some $\delta > 0$).

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The “big-O” notation hides arbitrary multiplicative / additive constants.

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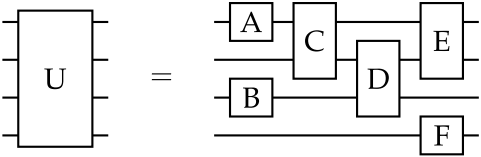

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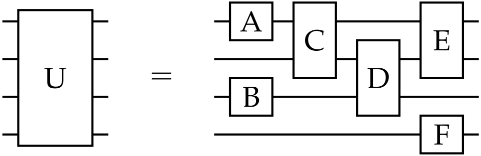

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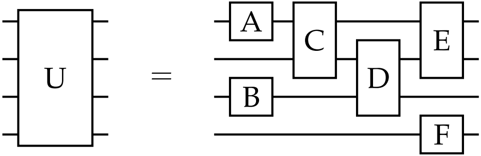

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- Sometimes it is reasonable to measure the complexity of the algorithms by the number of queries to the input used.

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Theorem [Shor '97]

There is a quantum algorithm which finds the prime factors of an n-digit integer in time $O(n^3)$.

Shor's algorithm: complexity comparison

Very roughly (ignoring constant factors!):

Number of digits	Timesteps (quantum)	Timesteps (classical)
100	10^6	$\sim 4 \times 10^5$
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- The fastest computer on the Top500 supercomputer list ($\sim 3.4 \times 10^{16}$ operations per second) in $\sim 1.2 \times 10^{17}$ years.

(see e.g. [Van Meter et al '05] for a more detailed comparison)

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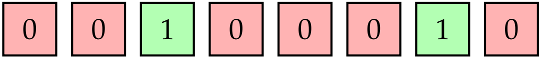

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0	0	1	0	0	0	1	0

- On a classical computer, this task could require 2^n queries to f in the worst case. But on a quantum computer, Grover's algorithm [Grover '97] can solve the problem with $O(\sqrt{2^n})$ queries to f (and bounded error).

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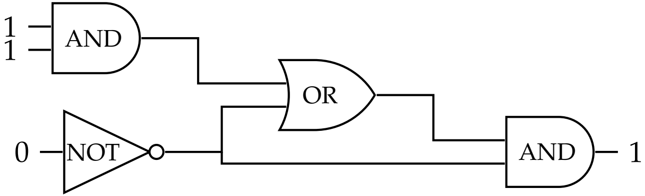

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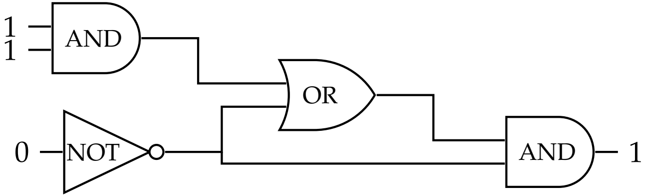

- Grover's algorithm improves the runtime from $O(2^n)$ to $O(2^{n/2})$: applications to design automation, circuit equivalence, model checking, …

Applications of Grover's algorithm

An important generalisation of Grover's algorithm is known as amplitude amplification.

Amplitude amplification **[Brassard et al '00]**

Assume we are given access to a "checking" function f, and a probabilistic algorithm $\mathcal{A}$ such that

$$\Pr[\mathcal{A} \text{ outputs } w \text{ such that } f(w) = 1] = \epsilon.$$

Then we can find w such that $f(w) = 1$ with $O(1/\sqrt{\epsilon})$ uses of f.

Gives a quadratic speed-up over classical algorithms based on the use of f as a black box.

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- Approximating the ℓ_1 distance between probability distributions on n elements in $O(\sqrt{n})$ time [Bravyi et al '09]
- ...

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Given a Hamiltonian H describing a physical system, and an initial state $|\psi_0\rangle$ of that system, produce the state

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- Applications: quantum chemistry, superconductivity, metamaterials, high-energy physics, … [Georgescu et al '13]

“Solving” linear equations

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Later improved to time $O(\kappa \log^3 \kappa \, \text{poly}(d) \log N)$ [Ambainis ’10].

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More recent applications of this algorithm include:

- "Solving" differential equations [Leyton and Osborne '08] [Berry '10]
- Data fitting [Wiebe et al '12]
- Space-efficient matrix inversion [Ta-Shma '13]

Quantum walks

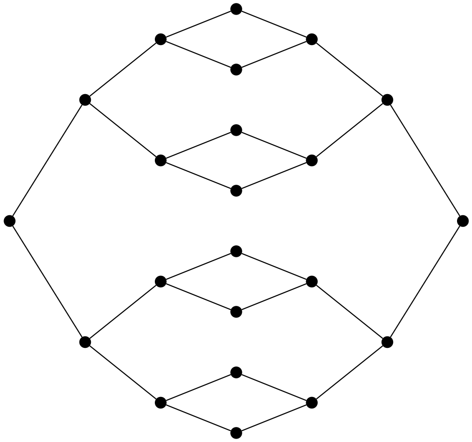
A quantum walk on a graph is a quantum generalisation of a classical random walk.

- A continuous-time quantum walk for time t on a graph with adjacency matrix A is the application of the unitary operator e^{-iAt}.

- Continuous-time quantum walks can be efficiently implemented as quantum circuits using Hamiltonian simulation.

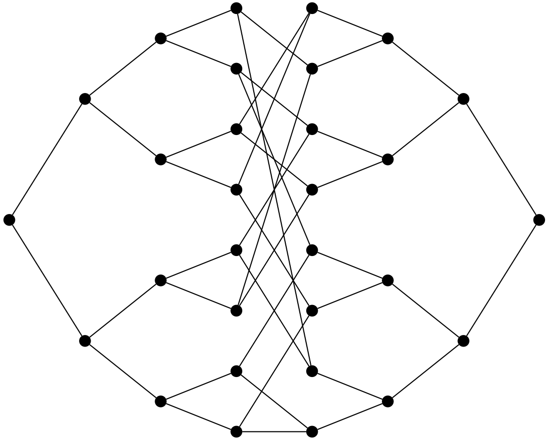
Quantum walks

Consider the graph formed by gluing two binary trees with N vertices together, e.g.:



Quantum walks

Now add a random cycle in the middle:



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Theorem [Childs et al '02]

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Other applications of continuous-time quantum walks:

- Spatial search [Childs and Goldstone '03]
- Evaluation of boolean formulae [Farhi et al '07] [Childs et al '07]

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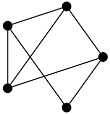
Element Distinctness can be solved using $O(n^{2/3})$ queries.

- The algorithm is based on discrete-time quantum walks.
- Time complexity is the same up to polylogarithmic factors.
- Generalisation to finding a k-subset of $\mathbb{Z}^n$ satisfying any property: uses $O(n^{k/(k+1)})$ queries.

Some examples

The same quantum walk framework lends itself to many different search problems, such as:

- Finding a triangle in a graph: $O(n^{1.3})$ queries, vs. classical $O(n^2)$ [Magniez et al '03] [Jeffery et al '12]



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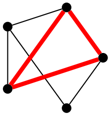

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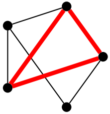

- Matrix product verification: $O(n^{5/3})$ queries, vs. classical $O(n^2)$ [Buhrman and Špalek '04]

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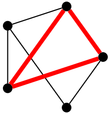

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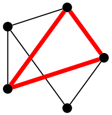

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- Testing group commutativity: $O(n^{2/3} \log n)$ queries, vs. classical $O(n)$ [Magniez and Nayak '05]

Yet more algorithms

There are a number of other quantum algorithms which I don't have time to go into:

- Hidden subgroup problems (e.g. [Bacon et al '05])
- Number-theoretic problems (e.g. [Fontein and Wocjan '11], . . .)
- Formula evaluation (e.g. [Reichardt and Špalek '07])
- Tensor contraction (e.g. [Arad and Landau '08])
- Hidden shift problems (e.g. [Gavinsky et al '11])
- Adiabatic optimisation (e.g. [Farhi et al '00])
- . . .

. . . as well as the entire field of quantum communication complexity.

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- Understanding multiple-prover quantum Merlin-Arthur proof systems has given new lower bounds on the classical complexity of computing tensor and matrix norms [Harrow and AM '10]
- New limitations on classical data structures, codes and formulas (see e.g. [Drucker and de Wolf '09])

Summary and further reading

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Some further reading:

- “Quantum algorithms for algebraic problems” [Childs and van Dam ’08]
- “Quantum walk based search algorithms” [Santha ’08]
- “Quantum algorithms” [Mosca ’08]
- “New developments in quantum algorithms” [Ambainis ’10]

Summary and further reading

There are many quantum algorithms, solving many different problems, using many different techniques.

Some further reading:

- "Quantum algorithms for algebraic problems" [Childs and van Dam '08]
- "Quantum walk based search algorithms" [Santha '08]
- "Quantum algorithms" [Mosca '08]
- "New developments in quantum algorithms" [Ambainis '10]

Thanks!

Primitive: Phase estimation

Phase estimation [Cleve et al '97] [Kitaev '95]

Given access to a unitary U and an eigenvector $|\psi\rangle$ such that $U|\psi\rangle = e^{2\pi i\phi}|\psi\rangle$, we can estimate ϕ up to additive error ϵ with $O(1/\epsilon)$ uses of U.

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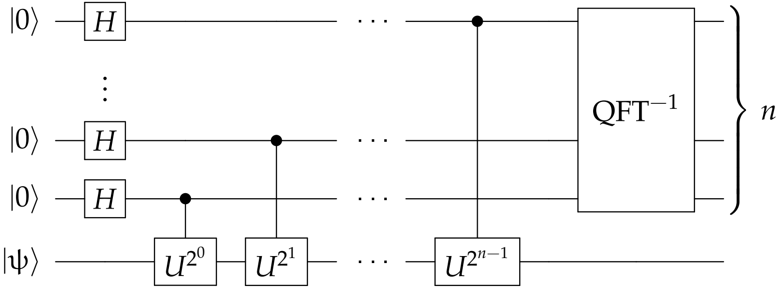
We apply the following circuit with $n = O(\log 1/\epsilon)$:



and then measure the first n qubits.