$2 + \epsilon$ quantum learning algorithms

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Talk based on joint work with Andris Ambainis and ongoing joint work with Scott Aaronson, David Chen, Daniel Gottesman and Vincent Liew.

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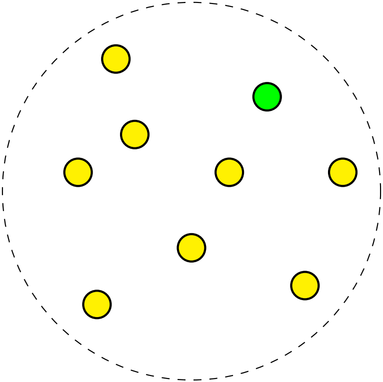
What is learning?



In this talk

Learning a set $S \equiv$ identifying an arbitrary, unknown object picked from S.

This talk

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On this principle, I'll talk about three optimal quantum algorithms for learning an unknown...

- ...bit-string, given access to "wildcard" queries;
- ...low-degree multilinear polynomial;
- ...stabilizer state.

Bonus mini-result: A composition theorem for classical decision tree complexity.

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Example

Imagine the hidden string is $x = 01101$. Then querying...

- $0 * 1 * 1$ returns 1;
- $*1 * 1*$ returns 0.

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Contrast: In the standard model, there is a quantum speed-up by about a factor of 2 [van Dam '98], and this is optimal.

Solving SWW

The solution to SWW is based on this claim:

Measurement Lemma

Fix $n \geqslant 1$ and, for any $0 \leqslant k \leqslant n$, set

$$|\psi_x^k\rangle := \frac{1}{\binom{n}{k}^{1/2}} \sum_{S \subseteq [n], |S| = k} |S\rangle |x_S\rangle,$$

where $|x_S\rangle := \bigotimes_{i \in S} |x_i\rangle$. Then, for any $k = n - O(\sqrt{n})$, there is a quantum measurement (POVM) which, on input $|\psi_x^k\rangle$, outputs $\widetilde{x}$ such that the expected Hamming distance $d(x, \widetilde{x})$ is $O(1)$.

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Why does this let us solve SWW?

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- ...but after each measurement, an expected $O(1)$ bits are incorrect.
- How to fix these?

Combinatorial group testing (CGT)

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In particular, we would like to minimise the dependence on n.

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- See the book “Combinatorial Group Testing and Its Applications” [Du and Hwang ’00] for more.

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- In the CGT problem, we have access to an oracle which computes $f(s) = \bigvee_i x_i s_i$ for arbitrary $s \in \{0,1\}^n$. But if $|x| \leqslant 1$, then for any s, $\bigvee_i x_i s_i = x \cdot s$.

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- Following each successful query, we reduce k by 1 and exclude the bit that we just learned from future queries.
- In order to learn x completely, the expected overall number of queries used is $O(k)$.

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- So we can use the algorithm for CGT to find, and correct, all incorrect bits using $O(1)$ queries.

Summary

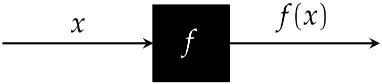
- Using an efficient algorithm for CGT as a subroutine, we can solve search with wildcards using $O(\sqrt{n})$ queries.

- This is a square-root speed-up which (apparently) does not come from amplitude amplification or quantum walks.

- Open problem: Determine the quantum query complexity of CGT. We have an upper bound of $O(k)$ and a lower bound of $\Omega(\sqrt{k})$.

Learning classical oracles

Consider the following basic problem.



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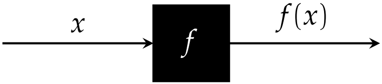

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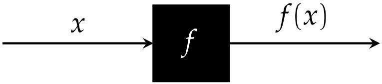

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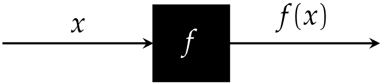

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- If f is arbitrary, we need q^n classical queries.
- If f is picked from a known set $\mathcal{F}$, we need at least $\log_q |\mathcal{F}|$ classical queries.
- On a quantum computer, we have the ability to query f in superposition, i.e. to perform the map

$$|x\rangle|z\rangle \mapsto |x\rangle|z + f(x)\rangle.$$

Learning multilinear polynomials

$f : \mathbb{F}_q^n \to \mathbb{F}_q$ is a degree d multilinear polynomial:

$$f(x) = \sum_{S \subseteq [n], |S| \leqslant d} \alpha_S \prod_{i \in S} x_i$$

for some coefficients $\alpha_S \in \mathbb{F}_q$, where $[n] := \{1, \ldots, n\}$.

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- Note that for $S = \emptyset$ we define $\prod_{i \in S} x_i = 1$.
- For example, any multilinear polynomial of degree 3 can be written as

$$f(x) = \alpha_\emptyset + \sum_i \alpha_{\{i\}} x_i + \sum_{i<j} \alpha_{\{i,j\}} x_i x_j + \sum_{i<j<k} \alpha_{\{i,j,k\}} x_i x_j x_k.$$

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- The set of degree d polynomials over $\mathbb{F}_2$ is known as the binary Reed-Muller code of order d.

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- Upper bound: It suffices to query $f(x)$ for all strings $x \in \mathbb{F}_q^n$ that contain only 0 and 1, and such that $|x| \leqslant d$.
- Lower bound: there are $q^{\Theta(n^d)}$ distinct multilinear degree d polynomials of n variables over $\mathbb{F}_q$; each classical query to f only provides $\log_2 q$ bits of information.

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Notes:

- The lower bound follows from Holevo's theorem.
- The Bernstein-Vazirani algorithm [Bernstein and Vazirani '97] is the case $q = 2$, $d = 1$.
- Rötteler previously gave a bounded-error quantum algorithm for the case $q = 2$, $d = 2$ [Rötteler '09].
- A quantum algorithm for estimating a quadratic form over the reals had previously been given by Jordan [Jordan '08].

The algorithm (sketch)

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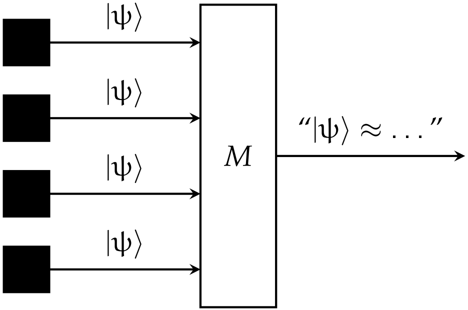
- So we can learn f using $O(n^{d-1})$ queries.

Learning quantum states

Consider the basic task of quantum state estimation.



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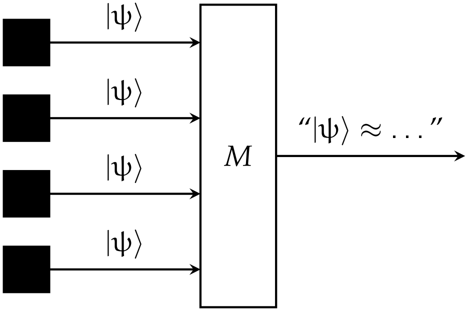

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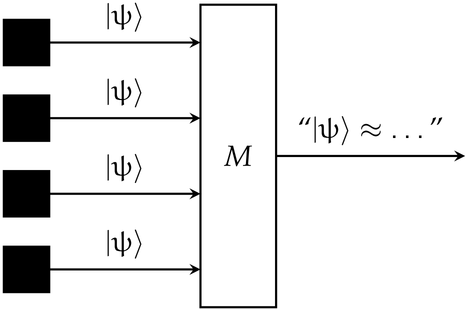

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- Standard quantum state tomography uses $2^{\Theta(n)}$ copies of $|\psi\rangle$ to achieve constant fidelity.
- Can we do better?

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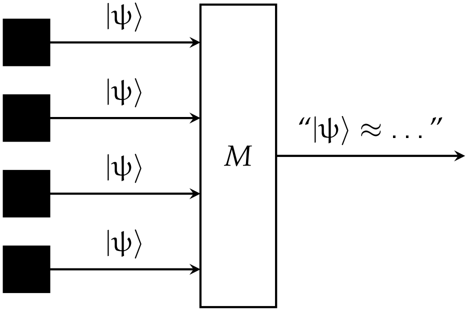

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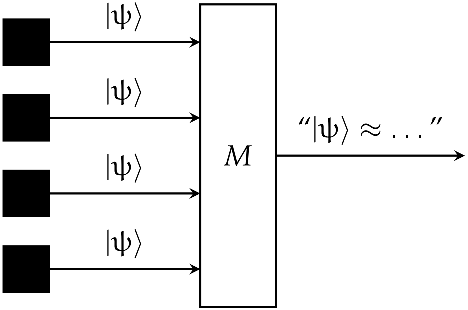

- To achieve constant fidelity between our guess and $|\psi\rangle$, we need $2^{\Omega(n)}$ copies of $|\psi\rangle$.
- In order to determine $|\psi\rangle$ efficiently (using $\mathrm{poly}(n)$ copies) we must restrict to classes of states which have efficient descriptions, or change the problem.

Learning quantum states

Some examples where this has been done:

- [Cramer et al '10] give an efficient algorithm for learning matrix product states.
- [Aaronson '06] introduces "pretty good tomography": relax to attempting to predict the outcomes of "most" measurements on the state.
- [Flammia and Liu '11] and [da Silva et al '11] give efficient algorithms for certifying the production of certain states.

Learning stabilizer states

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- $|\psi\rangle$ is a stabilizer state of n qubits if there exists a subgroup G of 2^n pairwise commuting Pauli matrices (with ± 1 phases) such that $M|\psi\rangle = |\psi\rangle$ for all $M \in G$.
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A stabilizer state of n qubits is completely specified by a generating set for its stabilizer (n Pauli matrices on n qubits). There are $2^{\Theta(n^2)}$ stabilizer states of n qubits.

Prior work on learning stabilizer states

[Aaronson and Gottesman '08] have previously given quantum algorithms for learning an unknown stabilizer state $|\psi\rangle$:

- An algorithm which uses $O(n)$ copies of $|\psi\rangle$ and runs in time $O(n^4)$;
- An algorithm which uses $O(n^2)$ copies of $|\psi\rangle$, runs in time $O(n^4)$ and uses only single-copy measurements.

Learning stabilizer states

Theorem

There is a quantum algorithm which learns an unknown stabilizer state $|\psi\rangle$ given access to $O(n)$ copies of $|\psi\rangle$, and runs in time $O(n^3)$ (or better).

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Notes on this result:

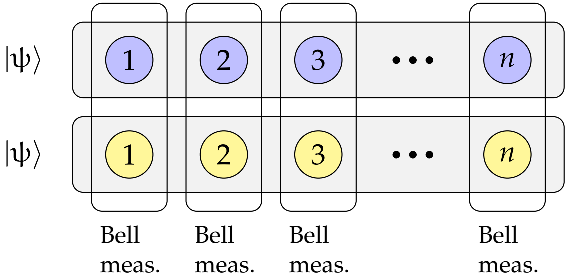
- By Holevo's theorem, this is optimal in terms of the scaling of the number of copies of $|\psi\rangle$ used.
- Any algorithm for learning stabilizer states requires $\Omega(n^2)$ time just to write the output.
- The algorithm makes measurements on two copies of $|\psi\rangle$ at a time.

The algorithm

The algorithm is based on the following subroutine.

Bell sampling

1. Create two copies of $|\psi\rangle$.
2. Measure each pair of qubits of $|\psi\rangle^{\otimes 2}$ in the Bell basis.



Learning stabilizer states

- For $z, x \in \{0, 1\}$, write $\sigma_{zx} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^x$.

- For $s \in \{0, 1\}^{2n}$, write

$$\sigma_s := \sigma_{s_1 s_2} \otimes \cdots \otimes \sigma_{s_{2n-1} s_{2n}}.$$

Fact

Let $|\psi\rangle$ be a state of n qubits. Performing Bell sampling on $|\psi\rangle^{\otimes 2}$ returns outcome s with probability

$$\frac{|\langle\psi|\sigma_s|\psi^*\rangle|^2}{2^n}.$$

Bell sampling and stabilizer states

- Up to an overall phase every stabilizer state $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \frac{1}{\sqrt{|A|}} \sum_{x \in A} i^{\ell(x)} (-1)^{q(x)} |x\rangle,$$

where A is an affine subspace of $\mathbb{F}_2^n$, and $\ell, q : \{0,1\}^n \to \{0,1\}$ are linear and quadratic (respectively) polynomials over $\mathbb{F}_2$ [Dehaene and Moor '02].

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- Hence

$$|\psi^*\rangle = \sigma_{10}^{\otimes S} |\psi\rangle.$$

Bell sampling and stabilizer states

- If we perform Bell sampling on $|\psi\rangle^{\otimes 2}$, we receive outcome t with probability

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- Bell sampling gives an outcome r which is uniformly distributed on the set $\{t \oplus s : t \in T\}$ for some $s \in \{0,1\}^{2n}$.

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  - In order to find a basis for T, we can therefore produce $k + 1$ Bell samples $r_0, r_1, \ldots, r_k$ and consider the uniformly random elements of T given by $r_1 \oplus r_0, r_2 \oplus r_0, \ldots, r_k \oplus r_0$.
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  - If the dimension of the subspace of $\mathbb{F}_2^{2n}$ spanned by these vectors is n, any basis of this subspace is a basis for T.
- Although T does not contain information about phases, determining T suffices to uniquely determine $|\psi\rangle$.
  - Once we have found a basis for T, we can measure $|\psi\rangle$ in the eigenbasis of each corresponding Pauli matrix M to decide whether $M|\psi\rangle = |\psi\rangle$ or $M|\psi\rangle = -|\psi\rangle$.

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5. For each element of B, measure a copy of $|\psi\rangle$ in the eigenbasis of the corresponding Pauli matrix M to determine whether $M|\psi\rangle = |\psi\rangle$ or $M|\psi\rangle = -|\psi\rangle$.

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- The algorithm fails (i.e. does not identify $|\psi\rangle$) if each of the $2n$ samples $r \oplus r_0$ lies in a subspace of T of dimension at most $n-1$. This occurs with probability at most 2^{-n}.

Bonus: a composition theorem for decision tree complexity

Imagine we want to compute a function of the form

$$h(x) = g(f^1(x^1), \dots, f^n(x^n)),$$

where $x^i \in \{0,1\}^{m_i}$, using the minimal number of classical queries to x.

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- Replace g with the function $\bar{g}$ given by substituting the values taken by any constant functions f^i into g.
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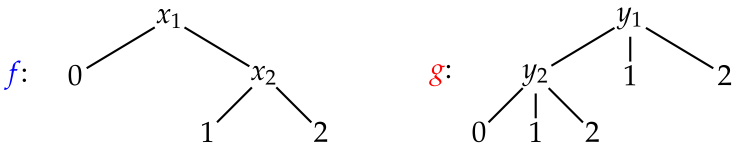
"Theorem": The x^i inputs are independent, so this is the most efficient way to compute g.

Counterexample to "theorem"

Let $f : \{0,1\}^2 \to \{0,1,2\}$ and $g : \{0,1,2\}^2 \to \{0,1,2\}$ be defined by the decision trees below (where edges correspond to elements of $\{0,1\}$ or $\{0,1,2\}$ in ascending order from left to right).



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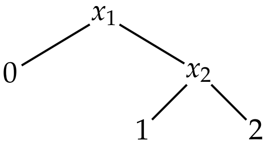

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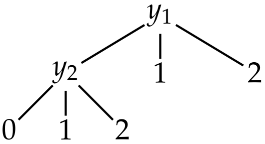
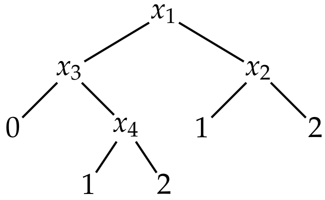


Set $h(x_1, x_2, x_3, x_4) = g(f(x_1, x_2), f(x_3, x_4))$. Then h can be computed using only 3 queries:



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- Implies various corollaries, e.g. a direct sum theorem for decision tree complexity (a special case of a result of [Jain, Klauck and Santha '10]) and optimal bounds for iteratively defined functions.
- The quantum equivalent of this result was proven by [Høyer, Lee and Špalek '07] and [Reichardt '09].

Summary

We can learn…

- …n-bit strings with $O(\sqrt{n})$ wildcard queries;
- …degree d n-variate multilinear polynomials with $O(n^{d-1})$ queries;
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Open problems:

- Determine the quantum query complexity of CGT.
- Other applications of SWW! A possible example: testing juntas.
- What about testing stabilizer states?

Thanks!

Some further reading:

- The algorithm for search with wildcards: **arXiv:1210.1148** (joint work with Andris Ambainis)
- The algorithm for learning multilinear polynomials: **arXiv:1105.3310**
- The algorithm for learning stabilizer states: **arXiv:13??.????** (joint work with Scott Aaronson, David Chen, Daniel Gottesman and Vincent Liew)
- The composition theorem for decision tree complexity: **arXiv:1302.4207**

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- D_k can be upper bounded using Fourier duality and some combinatorics.