Three quantum learning algorithms

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Talk based on joint work with Andris Ambainis and ongoing joint work with Scott Aaronson, David Chen, Daniel Gottesman and Vincent Liew.

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What is learning?

In this talk

Learning a set $S \equiv$ identifying an arbitrary, unknown object picked from $S$. 
A little learning is a dangerous thing; 
drink deep, or taste not the Pierian spring: 
there shallow draughts intoxicate the brain, 
and drinking largely sobers us again.

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This talk

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On this principle, I’ll talk about three optimal quantum algorithms for learning an unknown...

- …bit-string, given access to “wildcard” queries;
- …low-degree multilinear polynomial;
- …stabilizer state.

Bonus mini-result: A composition theorem for classical decision tree complexity.
Search with wildcards

- We are given access to an unknown $n$-bit string $x$. 

Example: Imagine the hidden string is $x = 01101$. Then querying...

- $0^*1^*1^*$ returns 1;
- $^*1^*^*$ returns 0.
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- Our task is to determine $x$ using the minimum expected number of queries.
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- Our task is to determine $x$ using the minimum expected number of queries.
- The different possible queries are given by strings $s \in \{0, 1, *\}^n$. A query $q_x(s)$ returns 1 if $x_i = s_i$ for all $i$ such that $s_i \neq *$, and returns 0 otherwise.
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- Our task is to determine \( x \) using the minimum expected number of queries.
- The different possible queries are given by strings \( s \in \{0, 1, \ast\}^n \). A query \( q_x(s) \) returns 1 if \( x_i = s_i \) for all \( i \) such that \( s_i \neq \ast \), and returns 0 otherwise.
- A generalisation of the simple “standard” model where each query is to an individual bit of \( x \).
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**Theorem**

Search with wildcards can be solved with $O(\sqrt{n})$ quantum queries on average.

**Contrast:** In the standard model, there is a quantum speed-up by about a factor of 2 [van Dam ’98], and this is optimal.
Solving SWW

The solution to SWW is based on this claim:

**Measurement Lemma**

Fix \( n \geq 1 \) and, for any \( 0 \leq k \leq n \), set

\[
|\psi^k_x\rangle := \frac{1}{\binom{n}{k}^{1/2}} \sum_{S \subseteq [n], |S|=k} |S\rangle|x_S\rangle
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where \(|x_S\rangle := \bigotimes_{i \in S} |x_i\rangle\). Then, for any \( k = n - O(\sqrt{n}) \), there is a quantum measurement (POVM) which, on input \(|\psi^k_x\rangle\), outputs \(\tilde{x}\) such that the expected Hamming distance \(d(x, \tilde{x})\) is \(O(1)\).
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This is surprising because the equivalent classical statement is not true!
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Why does this let us solve SWW?
Our algorithm for SWW repeatedly uses the lemma to learn $O(\sqrt{n})$ bits of $x$ at a time in superposition.
The measurement lemma $\Rightarrow$ solving SWW

- Our algorithm for SWW repeatedly uses the lemma to learn $O(\sqrt{n})$ bits of $x$ at a time in superposition.

- Imagine we have $|\psi_k^x\rangle$. For $k' > k$, this can be mapped to

$$\sum_{S':S' \subseteq [n], |S'|=k'} |S'\rangle \left( \sum_{S:S \subseteq S', |S|=k} |S\rangle |x_S\rangle \right) = \sum_{S:S \subseteq [n], |S|=k'} |S\rangle |\psi_{x_S}^k\rangle,$$

so if we can map $|\psi_{x_S}^k\rangle \mapsto |x_S\rangle$, we've made $|\psi_{x}^{k'}\rangle$. 
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- How to fix these?
Combinatorial group testing (CGT)

Proposed by [Dorfman ’43] as a means of “weeding out all syphilitic men called up for induction”.

The abstract problem is:

We have a set of $n$ items $x_1, \ldots, x_n \in \{0, 1\}$. Exactly $k \ll n$ items $x_i$ are special and have $x_i = 1$. We are allowed to query any subset $S \subseteq [n] := \{1, \ldots, n\}$. A query returns 1 if any items in $S$ are special. We want to output the identities of all of the special items using the minimal number of queries. In particular, we would like to minimise the dependence on $n$.
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Classical results

- The number of classical queries required to solve CGT is $\Theta(k \log(n/k))$.
  - Lower bound: information-theoretic argument.
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**The $k = 1$ case**

If $k = 1$, CGT can be solved exactly using one quantum query.
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Basic idea:

- To learn $x$, suffices to be able to compute the function $x \cdot s = \bigoplus_i x_i s_i$ for arbitrary $s \in \{0, 1\}^n$ (as with e.g. the quantum oracle interrogation algorithm of [van Dam ’98]).
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- In the CGT problem, we have access to an oracle which computes $f(s) = \bigvee_i x_is_i$ for arbitrary $s \in \{0, 1\}^n$. But if $|x| \leq 1$, then for any $s$, $\bigvee_i x_is_i = x \cdot s$. 
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- If $S$ contains exactly one 1 bit at position $i$, which will occur with probability at least $(1 - 1/k)^{k-1} \geq 1/e$, we are guaranteed to learn $i$. 

We can check whether the index $\tilde{i}$ we received really is a 1 by making one more query to index $\tilde{i}$. Following each successful query, we reduce $k$ by 1 and exclude the bit that we just learned from future queries. In order to learn $x$ completely, the expected overall number of queries used is $O(k)$. 
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A wildcard query corresponding to $S \subseteq [n]$ and the string $\tilde{x}_S$ returns 1 iff all bits of $\tilde{x}_S$ are correct. Negating the output gives a query that behaves the same as a CGT query.
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So we can use the algorithm for CGT to find, and correct, all incorrect bits using $O(1)$ queries.
Summary

- Using an efficient algorithm for CGT as a subroutine, we can solve search with wildcards using $O(\sqrt{n})$ queries.

- This is a square-root speed-up which (apparently) does not come from amplitude amplification or quantum walks.

- **Open problem:** Determine the quantum query complexity of CGT. We have an upper bound of $O(k)$ and a lower bound of $\Omega(\sqrt{k})$. 
Learning classical oracles

Consider the following basic problem.

We are given access to a function $f : \mathbb{F}_q^n \rightarrow \mathbb{F}_q$. We would like to identify $f$.

* If $f$ is arbitrary, we need $q^n$ classical queries.
* If $f$ is picked from a known set $F$, we need at least $\log_2 |F|$ queries.

We say that $F$ can be learned using $t$ queries if any function $f \in F$ can be identified with $t$ uses of $f$ (perhaps allowing some probability of error).
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Learning classical oracles on a quantum computer

On a quantum computer, we have the ability to query $f$ in superposition, i.e. to perform the map

$$|x⟩|z⟩ \mapsto |x⟩|z + f(x)⟩.$$
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- One of the oldest results in quantum computing: the Bernstein-Vazirani algorithm [Bernstein and Vazirani ’97].

**Theorem (Bernstein and Vazirani)**

The class of linear functions $f : \mathbb{F}_2^n \to \mathbb{F}_2$ can be learned with certainty using 1 quantum query.

$f$ is linear if $f(x + y) = f(x) + f(y)$; equivalently, $f(x) = \ell \cdot x$ for some $\ell \in \mathbb{F}_2^n$. 
Learning multilinear polynomials

$f : \mathbb{F}_q^n \rightarrow \mathbb{F}_q$ is a degree $d$ multilinear polynomial:

$$f(x) = \sum_{S \subseteq [n], |S| \leq d} \alpha_S \prod_{i \in S} x_i$$

for some coefficients $\alpha_S \in \mathbb{F}_q$, where $[n] := \{1, \ldots, n\}$.

- Note that for $S = \emptyset$ we define $\prod_{i \in S} x_i = 1$. 
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- Note that for $S = \emptyset$ we define $\prod_{i \in S} x_i = 1$.
- For example, any multilinear polynomial of degree 3 can be written as

\[
f(x) = \alpha_{\emptyset} + \sum_{i} \alpha_{\{i\}} x_i + \sum_{i < j} \alpha_{\{i,j\}} x_i x_j + \sum_{i < j < k} \alpha_{\{i,j,k\}} x_i x_j x_k.
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- In the important special case \( q = 2 \) (boolean functions), every polynomial is multilinear.
- The set of degree \( d \) polynomials over \( \mathbb{F}_2 \) is known as the binary Reed-Muller code of order \( d \).
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Fact
The class of degree $d$ multilinear polynomials in $n$ variables over $\mathbb{F}_q$ can be learned exactly using $O(n^d)$ classical queries, and this is optimal.
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- **Upper bound:** It suffices to query $f(x)$ for all strings $x \in \mathbb{F}_q^n$ that contain only 0 and 1, and such that $|x| \leq d$.

- **Lower bound:** there are $q^{\Theta(n^d)}$ distinct multilinear degree $d$ polynomials of $n$ variables over $\mathbb{F}_q$; each classical query to $f$ only provides $\log_2 q$ bits of information.
The class of degree $d$ multilinear polynomials in $n$ variables over $\mathbb{F}_q$ can be learned exactly using $O(n^{d-1})$ quantum queries, and this is optimal.

Notes:
- The lower bound follows from Holevo's theorem.
- The Bernstein-Vazirani algorithm is the case $q=2, d=1$.
- Rötteler previously gave a bounded-error quantum algorithm for the case $q=2, d=2$ [Rötteler '09].
- A quantum algorithm for estimating a quadratic form over the reals had previously been given by Jordan [Jordan '08].
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- A quantum algorithm for estimating a quadratic form over the reals had previously been given by Jordan [Jordan ’08].
The algorithm

The algorithm will be based on efficient learning of linear functions, via the following lemma [de Beaudrap et al ’02, van Dam et al ’02].

**Lemma 1**

Let $f : \mathbb{F}_q^n \rightarrow \mathbb{F}_q$ be linear, and let $g : \mathbb{F}_q^n \rightarrow \mathbb{F}_q$ be the function $g(x) = f(x) + \beta$ for some constant $\beta \in \mathbb{F}_q$. Then $f$ can be determined exactly using one quantum query to $g$. 
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**Proof:** query $f$ in superposition and use the QFT over $\mathbb{F}^n_q$. 

The algorithm

For $S \subseteq [n]$, $|S| = k$, define

$$f_S(x) = \sum_{\beta_1, \ldots, \beta_k \in \{0,1\}} (-1)^{k-\sum_{i=1}^{k} \beta_i} f \left( x + \sum_{j=1}^{k} \beta_j e_{S_j} \right).$$

Here $e_i$ is the $i$'th element in the standard basis for $\mathbb{F}_q^n$; the inner sum is over $\mathbb{F}_q^n$ and the outer sum is over $\mathbb{F}_q$. 
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- For example, if $S = \{1, 2\}$:

$$f_S(x) = f(x) - f(x + e_1) - f(x + e_2) + f(x + e_1 + e_2).$$
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- Define the discrete derivative of $f$ in direction $i \in [n]$ as

  $$(\Delta_i f)(x) := f(x + e_i) - f(x).$$

- Then $f_S(x) = (\Delta_{S_1} \Delta_{S_2} \ldots \Delta_{S_k} f)(x)$. 
The algorithm

We will be interested in querying \( f_S \) for sets \( S \) of size \( d - 1 \). In this case, we have the following characterisation for multilinear polynomials \( f \).

Lemma 2

Let \( f : \mathbb{F}_q^n \rightarrow \mathbb{F}_q \) be a multilinear polynomial of degree \( d \) with expansion

\[
f(x) = \sum_{T \subseteq [n], |T| \leq d} \alpha_T \prod_{i \in T} x_i.
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Then, for any \( S \) such that \( |S| = d - 1 \),

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f_S(x) = \alpha_S + \sum_{k \notin S} \alpha_{S \cup \{k\}} x_k.
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**Proof:** follows easily from expressing $f$ in terms of discrete derivatives.
Learning all the degree \(d\) terms

The algorithm

\[
\textbf{foreach } S \subseteq [n] \text{ such that } |S| = d - 1 \text{ do}
\]
\[
| \quad \text{Use one query to } f_S \text{ to learn } \alpha_{S \cup \{k\}}, \text{ for all } k \not\in S; \quad |
\]
end

Output the function \(f_d(x) = \sum_{S \subseteq [n], |S| = d} \alpha_S \prod_{i \in S} x_i;\)

Proof of correctness:
By Lemma 2, for any \(S\) such that \(|S| = d - 1\), knowledge of the degree 1 component of \(f_S\) is sufficient to determine \(\alpha_{S \cup \{k\}}\), for all \(k \not\in S\).

So knowing the degree 1 part of \(f_S\) for all \(S \subseteq [n]\) such that \(|S| = d - 1\) is sufficient to completely determine all degree \(d\) coefficients of \(f\).
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```latex
\textbf{foreach} \ S \subseteq [n] \text{ such that } |S| = d - 1 \ \textbf{do}
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\end{itemize}
\textbf{end}

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The algorithm

\begin{verbatim}
foreach $S \subseteq [n]$ such that $|S| = d - 1$ do
    Use one query to $f_S$ to learn $\alpha_{S \cup \{k\}}$, for all $k \not\in S$;
end
Output the function $f_d(x) = \sum_{S \subseteq [n], |S| = d} \alpha_S \prod_{i \in S} x_i$;
\end{verbatim}

Proof of correctness:

- By Lemma 1, for any $S$ with $|S| = d - 1$, the degree 1 component of $f_S$ can be determined with one quantum query to $f_S$.

- So the algorithm completely determines the degree $d$ component of $f$ using $\binom{n}{d-1}$ queries to $f_S$, each of which uses $2^{d-1}$ queries to $f$. 
Once the degree $d$ component of $f$ has been learned, $f$ can be reduced to a degree $d - 1$ polynomial by crossing out the degree $d$ part whenever the oracle for $f$ is called.
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Whenever the oracle is called on $x$, we subtract $f_d(x)$ from the result (where $f_d$ is the degree $d$ part of $f$), at no extra query cost.
Finishing up

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- Inductively, $f$ can be determined completely using

$$2^{d-1} \binom{n}{d-1} + 2^{d-2} \binom{n}{d-2} + \cdots + 2n + 1 + 1$$

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- The number of queries used is therefore $O(n^{d-1})$ for constant $d$. 
Learning quantum states

Consider the basic task of **quantum state estimation**.

Given the ability to produce copies of an unknown *n*-qubit quantum state $|\psi\rangle$, we would like to estimate $|\psi\rangle$. 

- Standard quantum state tomography uses $2^{\Theta(n)}$ copies of $|\psi\rangle$ to achieve constant fidelity.

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Learning quantum states

Consider the basic task of quantum state estimation.

To achieve constant fidelity between our guess and $|\psi\rangle$, we need $2^{\Omega(n)}$ copies of $|\psi\rangle$.

In order to determine $|\psi\rangle$ efficiently (using $\text{poly}(n)$ copies) we must restrict to classes of states which have efficient descriptions, or change the problem.
Learning quantum states

Some examples where this has been done:


- [Aaronson ’06] introduces “pretty good tomography”: relax to attempting to predict the outcomes of “most” measurements on the state.

Learning stabilizer states

Today I’ll talk about a learning algorithm for another important class of quantum states with efficient descriptions: stabilizer states.
Learning stabilizer states

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- $|\psi\rangle$ is a stabilizer state of $n$ qubits if there exists a subgroup $G$ of $2^n$ pairwise commuting Pauli matrices (with $\pm 1$ phases) such that $M|\psi\rangle = |\psi\rangle$ for all $M \in G$.

- Examples include GHZ states, cluster states, states occurring in quantum error-correcting codes, …
Learning stabilizer states

Today I’ll talk about a learning algorithm for another important class of quantum states with efficient descriptions: stabilizer states.

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A stabilizer state of $n$ qubits is completely specified by a generating set for its stabilizer ($n$ Pauli matrices on $n$ qubits). There are $2^{\Theta(n^2)}$ stabilizer states of $n$ qubits.
Prior work on learning stabilizer states

[Aaronson and Gottesman ‘08] have previously given quantum algorithms for learning an unknown stabilizer state $|\psi\rangle$:

- An algorithm which uses $O(n)$ copies of $|\psi\rangle$ and runs in time $O(n^4)$;
- An algorithm which uses $O(n^2)$ copies of $|\psi\rangle$, runs in time $O(n^4)$ and uses only single-copy measurements.
Learning stabilizer states

**Theorem**
There is a quantum algorithm which learns an unknown stabilizer state $|\psi\rangle$ given access to $O(n)$ copies of $|\psi\rangle$, and runs in time $O(n^3)$ (or better).

Notes on this result:
By Holevo's theorem, this is optimal in terms of the scaling of the number of copies of $|\psi\rangle$ used. Any algorithm for learning stabilizer states requires $\Omega(n^2)$ time just to write the output. The algorithm makes measurements on two copies of $|\psi\rangle$ at a time.
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- Any algorithm for learning stabilizer states requires $\Omega(n^2)$ time just to write the output.

- The algorithm makes measurements on two copies of $|\psi\rangle$ at a time.
The algorithm

The algorithm is based on the following subroutine.

**Bell sampling**

1. Create two copies of $|\psi\rangle$.
2. Measure each pair of qubits of $|\psi\rangle^{\otimes 2}$ in the Bell basis.

\[
|\psi\rangle \quad 1 \quad 2 \quad 3 \quad \cdots \quad n
\]

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Learning stabilizer states

For \( z, x \in \{0, 1\} \), write \( \sigma_{zx} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^z \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^x \).

For \( s \in \{0, 1\}^{2n} \), write
\[
\sigma_s := \sigma_{s_1s_2} \otimes \cdots \otimes \sigma_{s_{2n-1}s_{2n}}.
\]

Fact

Let \( |\psi\rangle \) be a state of \( n \) qubits. Performing Bell sampling on \( |\psi\rangle \otimes^2 \) returns outcome \( s \) with probability
\[
\frac{|\langle \psi| \sigma_s |\psi^*\rangle|^2}{2^n}.
\]
Bell sampling and stabilizer states

- Up to an overall phase every stabilizer state $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \frac{1}{\sqrt{|A|}} \sum_{x \in A} i^{\ell(x)}(-1)^{q(x)}|x\rangle,$$

where $A$ is an affine subspace of $\mathbb{F}_2^n$, and $\ell, q : \{0, 1\}^n \rightarrow \{0, 1\}$ are linear and quadratic (respectively) polynomials over $\mathbb{F}_2$ [Dehaene and Moor ’02].
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- So $(-1)^{\ell(x)} = \prod_{i \in S} (-1)^{x_i}$ for some $S \subseteq [n]$. 
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- Hence

$$|\psi^*\rangle = \sigma_{10}^{\otimes S} |\psi\rangle.$$
Bell sampling and stabilizer states

If we perform Bell sampling on $|\psi\rangle \otimes^2$, we receive outcome $t$ with probability

$$\frac{|\langle \psi | \sigma_t | \psi^* \rangle|^2}{2^n} = \frac{|\langle \psi | \sigma_t \sigma_1^{S} | \psi \rangle|^2}{2^n}.$$
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$$\frac{|\langle \psi | \sigma_t | \psi^* \rangle|^2}{2^n} = \frac{|\langle \psi | \sigma_t \sigma_1 \otimes^S | \psi \rangle|^2}{2^n}.$$ 

- Let $G$ stabilize $|\psi\rangle$ and let $T$ denote the set of strings $t \in \{0,1\}^{2n}$ such that $\sigma_t \in G$, up to a phase. Then $T$ is an $n$-dimensional linear subspace of $\mathbb{F}_2^{2n}$. 

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Bell sampling gives an outcome $r$ which is uniformly distributed on the set $\{t \oplus s : t \in T\}$ for some $s \in \{0, 1\}^{2n}$. 

For any two such outcomes $r_1, r_2$, the sum $r_1 \oplus r_2$ is uniformly distributed in $T$. 

Although $T$ does not contain information about phases, determining $T$ suffices to uniquely determine $|\psi\rangle$. Once we have found a basis for $T$, we can measure $|\psi\rangle$ in the eigenbasis of each corresponding Pauli matrix $M$ to decide whether $M|\psi\rangle = |\psi\rangle$ or $M|\psi\rangle = -|\psi\rangle$. 

Bell sampling and stabilizer states
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In order to find a basis for $T$, we can therefore produce $k + 1$ Bell samples $r_0, r_1, \ldots, r_k$ and consider the uniformly random elements of $T$ given by $r_1 \oplus r_0, r_2 \oplus r_0, \ldots, r_k \oplus r_0$.

If the dimension of the subspace of $\mathbb{F}_2^{2n}$ spanned by these vectors is $n$, any basis of this subspace is a basis for $T$. 

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**Learning stabilizer states**

### The algorithm

1. Set $S = \emptyset$.  

2. Create two copies of $|\psi\rangle$ and perform Bell sampling, obtaining outcome $r_0$.  

3. Repeat the following $2^n$ times:  
   1. Create two copies of $|\psi\rangle$ and perform Bell sampling, obtaining outcome $r$.  
   2. Add $r \oplus r_0$ to $S$.  

4. Determine a basis for $S$; call this basis $B$.  

5. For each element of $B$, measure a copy of $|\psi\rangle$ in the eigenbasis of the corresponding Pauli matrix $M$ to determine whether $M|\psi\rangle = |\psi\rangle$ or $M|\psi\rangle = -|\psi\rangle$.  

## Learning stabilizer states

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### Detailed Steps:

- **Step 1:** Set $S = \emptyset$.
- **Step 2:** Create two copies of $|\psi\rangle$ and perform Bell sampling, obtaining outcome $r_0$. Then, add $r_\oplus r_0$ to $S$.
- **Step 3:** Determine a basis for $S$; call this basis $B$.
- **Step 4:** For each element of $B$, measure a copy of $|\psi\rangle$ in the eigenbasis of the corresponding Pauli matrix to determine whether $M|\psi\rangle = |\psi\rangle$ or $M|\psi\rangle = -|\psi\rangle$. 


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Summary of learning stabilizer states

- The algorithm uses $O(n)$ copies of $|\psi\rangle$. Time complexity is dominated by finding a basis for $S$ ($O(n^3)$ time or better).

- The algorithm fails (i.e. does not identify $|\psi\rangle$) if each of $2^n$ samples $r \oplus r_0$ lies in a subspace of $T$ of dimension at most $n-1$. This occurs with probability at most $2^{-n}$. 
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Imagine we want to compute a function of the form

\[ h(x) = g(f^1(x^1), \ldots, f^n(x^n)), \]

where \( x^i \in \{0, 1\}^{m_i} \), using the minimal number of classical queries to \( x \).
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One strategy to compute \( h \):

- Replace \( g \) with the function \( \bar{g} \) given by substituting the values taken by any constant functions \( f^i \) into \( g \).
- Compute \( \bar{g} \) using efficient algorithms for \( f^1, \ldots, f^n \) as black boxes.
**Bonus: a composition theorem for decision tree complexity**

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- Compute $\bar{g}$ using efficient algorithms for $f^1, \ldots, f^n$ as black boxes.

"Theorem": The $x^i$ inputs are independent, so this is the most efficient way to compute $g$. 
Counterexample to “theorem”

Let $f : \{0, 1\}^2 \to \{0, 1, 2\}$ and $g : \{0, 1, 2\}^2 \to \{0, 1, 2\}$ be defined by the decision trees below (where edges correspond to elements of $\{0, 1\}$ or $\{0, 1, 2\}$ in ascending order from left to right).

\[ f : \quad \begin{array}{c}
\text{0} \\
\text{x1} \\
\text{x2} \\
\text{1} \\
\text{2}
\end{array} \quad \begin{array}{c}
\text{g :} \\
\text{y1} \\
\text{y2} \\
\text{1} \\
\text{2}
\end{array} \]
Counterexample to “theorem”

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Set $h(x_1, x_2, x_3, x_4) = g(f(x_1, x_2), f(x_3, x_4))$. Then $h$ can be computed using only 3 queries:
Nevertheless...

**Theorem**

The above algorithm is optimal when \( \text{range}(f^i) \subseteq \{0, 1\} \) for all \( i \).
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Some notes on this result:

- Also holds for computing partial functions and relations.
- Implies various corollaries, e.g. a direct sum theorem for decision tree complexity (a special case of a result of [Jain, Klauck and Santha '10]) and optimal bounds for iteratively defined functions.
- The quantum equivalent of this result was proven by [Høyer, Lee and Špalek '07] and [Reichardt '09].
Summary

We can learn...

- ...n-bit strings with $O(\sqrt{n})$ wildcard queries;

- ...degree $d$ $n$-variate multilinear polynomials with $O(n^{d-1})$ queries;

- ...$n$-qubit stabilizer states with $O(n)$ copies.
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Open problems:

- Determine the quantum query complexity of CGT.
- Other applications of SWW! A possible example: testing juntas.
- What about testing stabilizer states?
Thanks!

Some further reading:

- The algorithm for search with wildcards: arXiv:1210.1148 (joint work with Andris Ambainis)


- The composition theorem for decision tree complexity: arXiv:1302.4207
Proving the measurement lemma

We finally need to prove we can distinguish the $|\psi^k_x\rangle$ states. We use the pretty good measurement (PGM).
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**Lemma**

The probability that the PGM outputs $y$ on input $|\psi^k_x\rangle$ is precisely $(\sqrt{G})_{xy}^2$, where

$$G_{xy} = \langle \psi^k_x | \psi^k_y \rangle = \frac{1}{\binom{n}{k}} \sum_{S \subseteq [n], |S| = k} [x_S = y_S] = \frac{(n-d(x,y))}{\binom{n}{k}}.$$
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We want to bound $D_k := \sum_{y \in \{0,1\}^n} d(x, y) (\sqrt{G}_{xy})^2$. 

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- We want to bound $D_k := \sum_{y \in \{0,1\}^n} d(x, y)(\sqrt{G_{xy}})^2$.
- $G_{xy}$ depends only on $x \oplus y$, so $G$ is diagonalised by the Fourier transform over $\mathbb{Z}_2^n$ and $D_k$ does not depend on $x$. 
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- $G_{xy}$ depends only on $x \oplus y$, so $G$ is diagonalised by the Fourier transform over $\mathbb{Z}_2^n$ and $D_k$ does not depend on $x$.
- $D_k$ can be upper bounded using Fourier duality and some combinatorics.