Complexity classification of local Hamiltonian problems

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Joint work with Toby Cubitt:
Introduction

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- The 3-SAT problem: given a boolean formula in conjunctive normal form with at most 3 variables per clause, is there a satisfying assignment to the formula?

\[(x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2) \land (x_4)\]
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- Solving 3-term linear equations: given a system of linear equations over \(\mathbb{F}_2\) with at most 3 variables per equation, is there a solution to all the equations?

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The first of these is NP-complete, the second is in P.
General constraint satisfaction problems

A very general way to study these kind of problems is via the framework of the problem $S$-CSP.

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- An instance of $S$-$CSP$ on $n$ bits is specified by a sequence of constraints picked from $S$ applied to subsets of the bits.
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The complexity of the $S$-CSP problem depends on the set $S$. 
A remarkable theorem of Schaefer allows this complexity to be completely characterised.

**Theorem [Schaefer ’78]**

$S$-CSP is either in P or NP-complete. Further, which of these is the case can be determined easily for a given $S$. 

This result has since been improved in a number of directions. In particular, [Creignou ’95] and [Khanna, Sudan and Williamson ’97] have completely characterised the complexity of the maximisation problem $k$-Max-CSP for boolean constraints. Here we are again given a system of constraints, but the goal is to maximise the number of constraints we can satisfy. An example problem of this kind is MAX-CUT.
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Local Hamiltonian problems

The natural quantum generalisation of CSPs is called $k$-local Hamiltonian [Kitaev, Shen and Vyalyi ’02].

- A $k$-local Hamiltonian is a Hermitian matrix $H$ on the space of $n$ qubits which can be written as

$$H = \sum_i H^{(i)},$$

where each $H^{(i)}$ acts non-trivially on at most $k$ qubits.
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\textbf{\textit{k-local Hamiltonian}}

We are given a \textit{k-local} Hamiltonian $H = \sum_{i=1}^{m} H^{(i)}$ on $n$ qubits, and two numbers $a < b$ such that $b - a \geq 1/\text{poly}(n)$. Promised that the smallest eigenvalue of $H$ is either at most $a$, or at least $b$, our task is to determine which of these is the case.

NB: we assume throughout that all parameters are “reasonable” (e.g. rational, polynomial in $n$).
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- 1-local Hamiltonian is in P, so is this the end of the line?
**k-local Hamiltonian** and condensed-matter physics

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A major motivation for this area is applications to physics.

- One of the most important themes in condensed-matter physics is calculating the ground-state energies of physical systems; this is essentially an instance of $k$-LOCAL HAMILTONIAN.
- For example, the (general) Ising model corresponds to the problem of finding the lowest eigenvalue of a Hamiltonian of the form

$$H = \sum_{i<j} \alpha_{ij} Z_i Z_j.$$ 

- This connection to physics motivates the study of $k$-LOCAL HAMILTONIAN with restricted types of interactions.
- The aim: to prove QMA-hardness of problems of direct physical interest.
Previously known results

A number of special cases of $k$-local Hamiltonian have previously been shown to be QMA-complete, e.g.:

- [Schuch and Verstraete ’09]:

$$H = \sum_{(i,j) \in E} X_i X_j + Y_i Y_j + Z_i Z_j + \sum_{k} \alpha_k X_k + \beta_k Y_k + \gamma_k Z_k,$$

where $E$ is the set of edges of a 2-dimensional square lattice;
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- [Biamonte and Love ’08]:

$$H = \sum_{i<j} J_{ij}X_iX_j + K_{ij}Z_iZ_j + \sum_{\alpha} \alpha_k X_k + \beta_k Z_k,$$

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$$H = \sum_{i<j} J_{ij}X_iZ_j + K_{ij}Z_iX_j + \sum_{\alpha} \alpha_k X_k + \beta_k Z_k.$$
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- As AM is in the polynomial hierarchy, it is considered unlikely that *k*-LOCAL HAMILTONIAN with stoquastic Hamiltonians is QMA-complete.
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- As AM is in the polynomial hierarchy, it is considered unlikely that $k$-LOCAL HAMILTONIAN with stoquastic Hamiltonians is QMA-complete.

- Later sharpened by [Bravyi, Bessen and Terhal '06], who showed that this problem is StoqMA-complete, where StoqMA is a complexity class between MA and AM.
The $S$-Hamiltonian problem

Let $S$ be a fixed subset of Hermitian matrices on at most $k$ qubits, for some constant $k$.

**$S$-Hamiltonian**

$S$-Hamiltonian is the special case of $k$-local Hamiltonian where the overall Hamiltonian $H$ is specified by a sum of matrices $H_i$, each of which acts non-trivially on at most $k$ qubits, and whose non-trivial part is proportional to a matrix picked from $S$. 
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We then have the following general question:

**Problem**

Given $S$, characterise the computational complexity of $S$-Hamiltonian.
Some examples

The $S$-HAMILTONIAN problem encapsulates many much-studied problems in physics. For example:

- The (general) Ising model:

\[
H = \sum_{i<j} \alpha_{ij} Z_i Z_j.
\]

For us this is the problem \{ZZ\}-HAMILTONIAN; it is known to be \text{NP-complete}. 

The (general) Ising model with transverse magnetic fields:

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H = \sum_{i<j} \alpha_{ij} Z_i Z_j + \sum \beta_k X_k.
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- The (general) Heisenberg model:

\[ H = \sum_{i<j} \alpha_{ij}(X_i X_j + Y_i Y_j + Z_i Z_j). \]

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We use “general” in the titles to emphasise that there is no implied spatial locality or underlying interaction graph.
Remarks on the problem

We assume that, given a set of interactions $S$, we are allowed to produce an overall Hamiltonian by applying each interaction $M \in S$ scaled by an arbitrary real weight, which can be either positive or negative.
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- Some of the interactions in $S$ could be non-symmetric under permutation of the qubits on which they act. We assume that we are allowed to apply such interactions to any permutation of the qubits.

- We can assume without loss of generality that the identity matrix $I \in S$ (we can add an arbitrary “energy shift”).
Allowing local terms

One variant of this framework is to allow arbitrary local terms ("magnetic fields").

**S-Hamiltonian with local terms**

*S-Hamiltonian with local terms* is the special case of *S-Hamiltonian* where *S* is assumed to contain *X, Y, Z*.

- This is equivalent to *S* containing all 1-local interactions.
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- This is equivalent to S containing all 1-local interactions.
- For any S, \textit{S-Hamiltonian with local terms} is at least as hard as \textit{S-Hamiltonian}. 

It is known that \textit{S-Hamiltonian with local terms} is QMA-complete when:

\begin{itemize}
  \item \quad S = \{XX + YY + ZZ\}\quad \text{[Schuch and Verstraete '09]}
  \item \quad S = \{XX, ZZ\}
  \item \quad S = \{XZ\}\quad \text{[Biamonte and Love '08]}
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- *S* = \{XX + YY + ZZ\} [Schuch and Verstraete ’09]
- *S* = \{XX, ZZ\} or *S* = \{XZ\} [Biamonte and Love ’08]
Our first result

Let $S$ be a fixed subset of Hermitian matrices on at most $k$ qubits, for some constant $k$.

**Theorem**

Let $S'$ be the subset formed by removing all 1-local terms from each element of $S$, and then deleting all 0-local matrices. Then:

1. If $S'$ is empty, $S$-HAMILTONIAN with local terms is in \( \mathbf{P} \);
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2. Otherwise, if there exists $U \in SU(2)$ such that $U$ locally diagonalises $S'$, then $S$-HAMILTONIAN with local terms is poly-time equivalent to the transverse Ising model;
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3. Otherwise, $S$-HAMILTONIAN WITH LOCAL TERMS is QMA-complete.
Explaining the second case

The second case is stated in terms of “local diagonalisation”:

- Let $M$ be a $k$-qubit Hermitian matrix.
- We say that $U \in SU(2)$ locally diagonalises $M$ if $U^\otimes k M (U^\dagger)^\otimes k$ is diagonal.
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- We say that $U \in SU(2)$ \textbf{locally diagonalises} $M$ if $U \otimes^k M (U^\dagger) \otimes^k$ is diagonal.
- We say that $U$ locally diagonalises $S$ if $U$ locally diagonalises $M$ for all $M \in S$.
- Note that matrices in $S$ may be of different sizes.

This case is poly-time equivalent to the \textbf{transverse Ising model $\{ZZ, X\}$-HAMILTONIAN}, i.e. Hamiltonians of the form

$$H = \sum_{i<j} \alpha_{ij} Z_i Z_j + \sum_k \beta_k X_k.$$ 

What is the complexity of solving this model?
The complexity of the transverse Ising model

- The problem is clearly \textbf{NP-hard}, by taking the weights $\beta_k$ of the $X$ terms to be 0.
The complexity of the transverse Ising model

• The problem is clearly NP-hard, by taking the weights $\beta_k$ of the $X$ terms to be 0.

• By conjugating any transverse Ising model Hamiltonian by local $Z$ operations on each qubit $k$ such that $\beta_k > 0$, which maps $X \mapsto -X$ and does not change the eigenvalues, we can assume $\beta_k \leq 0$. 
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- The resulting Hamiltonian is **stoquastic**, so $\{ZZ, X\}$-**HAMILTONIAN $\in$ StoqMA**.
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- We have not been able to characterise the complexity of this problem more precisely, so encapsulate it in a new complexity class \textbf{TIM}, where \textbf{NP} $\subseteq$ \textbf{TIM} $\subseteq$ \textbf{StoqMA}. 
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- **Future work**: the Transverse Ordered Boson Ynteraction and Anisotropic Symmetric Hamiltonians with Local Extensive Ynteractions...
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**Theorem**

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4. Otherwise, $S$-HAMILTONIAN is $\text{QMA-complete}$. 
In particular, we have that:

- The (general) **Heisenberg model** is QMA-complete 
  \((\mathcal{S} = \{XX + YY + ZZ\})\)
- The (general) **XY model** is QMA-complete \((\mathcal{S} = \{XX + YY\})\)

... as well as many other cases.

We can think of this result as a quantum analogue of **Schaefer’s dichotomy theorem**.
Proof techniques

We follow the standard pattern for proving dichotomy-type theorems:

“Isolate some special cases and prove that they are easy, then prove that everything else is hard.”
Proof techniques

We follow the standard pattern for proving dichotomy-type theorems:

“Isolate some special cases and prove that they are easy, then prove that everything else is hard.”

- The two results are proven using (fairly) different techniques, but both are based on reductions, rather than direct proofs using clock constructions or similar.

- The starting point for both is a normal form for 2-qubit Hermitian matrices.
The normal form

We use a very similar normal form to one identified by [Dür et al. ’01, Bennett et al. ’02]. An important special case:

**Lemma**

Let $H$ be a 2-qubit interaction which is symmetric under swapping qubits. Then there exists $U \in SU(2)$ such that the 2-local part of $U \otimes^2 H (U^\dagger) \otimes^2$ is of the form

$$\alpha XX + \beta YY + \gamma ZZ.$$
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Why is this useful? If we conjugate each term by $U \otimes^2$ in a 2-local Hamiltonian with only $H$ interactions, it doesn’t change the eigenvalues:

$$\sum_{i \neq j} \alpha_{ij} (U \otimes^2 H(U^\dagger) \otimes^2)_{ij} = U \otimes^n \left( \sum_{i \neq j} \alpha_{ij} H_{ij} \right) (U^\dagger) \otimes^n.$$
The next step

The basic idea:

“To prove QMA-hardness of $A$-Hamiltonian, approximately simulate some other set of interactions $B$, where $B$-HAMILTONIAN is QMA-hard.”
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- The first-order perturbative gadgets we use are based on ideas going back to [Oliveira and Terhal ’08] and [Schuch and Verstraete ’08].
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The basic idea:

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To prove QMA-hardness of $A$-Hamiltonian, approximately simulate some other set of interactions $B$, where $B$-HAMILTONIAN is QMA-hard.

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- To do this, we use two kinds of reductions, both based on perturbation theory.
- The first-order perturbative gadgets we use are based on ideas going back to [Oliveira and Terhal ’08] and [Schuch and Verstraete ’08].
- The basic idea: to implement an effective interaction across two qubits $a$ and $c$, add a new mediator qubit $b$ interacting with each of $a$ and $c$, and put a strong 1-local interaction on $b$. 
Claim (similar to results of [Schuch and Verstraete ’08])

For any $\gamma \neq 0$, $\{XX + \gamma ZZ\}$-HAMILTONIAN WITH LOCAL TERMS is QMA-complete.
Example

Claim (similar to results of [Schuch and Verstraete ’08])

For any $\gamma \neq 0$, $\{XX + \gamma ZZ\}$-Hamiltonian with local terms is QMA-complete.

We use the following perturbative gadget, taking $\Delta$ to be a large coefficient:

\[ \begin{align*}
XX + \gamma ZZ & \quad \Delta |1\rangle \langle 1| \\
a & \quad b & \quad c
\end{align*} \]
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  ![Diagram](attachment:diagram.png)

  This forces qubit $b$ to (approximately) be in the state $|0\rangle$. 

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\hline
a & b & c
\end{array}
\]

- This forces qubit $b$ to (approximately) be in the state $|0\rangle$.

- It turns out that, up to local and lower-order terms, the effective interaction across the remaining qubits is

\[H_{\text{eff}} \propto X_a X_c.\]
Example

- So, given access to terms of the form $XX + \gamma ZZ$, we can effectively make $XX$ terms. By subtracting from $XX + \gamma ZZ$, we can also make $ZZ$ terms.
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We can similarly show that:

- For any $\beta, \gamma \neq 0$, $\{XX + \beta YY + \gamma ZZ\}$-Hamiltonian with local terms is QMA-complete.
- $\{XZ - ZX\}$-Hamiltonian with local terms is QMA-complete.
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This turns out to be all the cases we need to complete the characterisation of $S$-HAMILTONIAN with local terms!
Recap: Our second result

Let $S$ be an arbitrary fixed subset of Hermitian matrices on at most 2 qubits.

**Theorem**

1. If every matrix in $S$ is 1-local, $S$-HAMILTONIAN is in $P$;
2. Otherwise, if there exists $U \in SU(2)$ such that $U$ locally diagonalises $S$, then $S$-HAMILTONIAN is NP-complete;
3. Otherwise, if there exists $U \in SU(2)$ such that, for each 2-qubit matrix $H_i \in S$, $U^\otimes 2 H_i (U^\dagger)^\otimes 2 = \alpha_i Z^\otimes 2 + A_i I + IB_i$, where $\alpha_i \in \mathbb{R}$ and $A_i, B_i$ are arbitrary single-qubit Hermitian matrices, then $S$-HAMILTONIAN is TIM-complete;
4. Otherwise, $S$-HAMILTONIAN is QMA-complete.
The easier cases

Cases (1) and (2) are the easiest:

1. The minimal eigenvalue of a sum of 1-local terms is the sum of the minimal eigenvalues.
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Case (3) is clearly no harder than $S$-HAMILTONIAN with local terms, so is contained in TIM; TIM-completeness follows by a reduction from $\{ZZ\}$-HAMILTONIAN with local terms.
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The most interesting case is (4)…
Proof techniques

If we do not have access to arbitrary 1-local terms, we can no longer use the same perturbative gadgets, so we rely on a different (and in some sense simpler) technique.
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- The basic idea: encode interactions within a subspace.
- Given two Hamiltonians $H$ and $V$, we form $\tilde{H} = V + \Delta H$, where $\Delta$ is a large parameter.
- Then $\tilde{H}_{<\Delta/2}$, the low-energy part of $\tilde{H}$, is effectively the same as $V_-$, the projection of $V$ onto the lowest-energy eigenspace of $H$. 
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**Projection Lemma (informal, based on [Oliveira+Terhal ’08])**

If $\Delta = \delta \|V\|^2$, then

$$\|\tilde{H}_{<\Delta/2} - V_-\| = O(1/\delta).$$
Example: the Heisenberg model

The case \( \mathcal{S} = \{XX + YY + ZZ\} \) illustrates the difficulties that we face when we do not have access to all 1-local terms. Let

\[
H = \sum_{i<j} \alpha_{ij}(X_iX_j + Y_iY_j + Z_iZ_j).
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- So the eigenspaces of $H$ are all invariant under conjugation by $U \otimes^n$!

This means that we cannot hope to implement an arbitrary Hamiltonian using only this interaction.
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Just as with classical CSPs, the way round this is to use encodings.
Example: the Heisenberg model

- We would like to find a gadget that encodes qubits, and lets us encode operations across qubits.
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- We try to encode a logical qubit within a triangle of 3 physical qubits:

- This is inspired by previous work on universality of the exchange interaction [Kempe et al. '00].
Example: the Heisenberg model

The Heisenberg interaction is equivalent to the swap (flip) operation

\[ F = \frac{1}{2}(I + XX + YY + ZZ). \]
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- The first step: decompose the three qubits (labelled 1-3) into the 4-dim **symmetric subspace** \( S_1 \) of 3 qubits and its orthogonal complement \( S_2 \).
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- On \( S_1 \), \( F \) acts as the identity. On \( S_2 \), with respect to the right basis we have

\[ F_{12} + F_{13} + F_{23} = 0, \quad -F_{12} = Z \otimes I, \quad \frac{1}{\sqrt{3}} (F_{13} - F_{23}) = X \otimes I. \]
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- By applying strong \( F \) interactions across all pairs of qubits, we can effectively project onto \( S_2 \).
- Then we can apply \( Z \) and \( X \) on two logical pseudo-qubits.
Example: the Heisenberg model

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Let the logical qubits in the first (resp. second) triangle be labelled (1,2) (resp. (3,4)).

It turns out that, by applying suitable linear combinations across qubits, we can effectively make

$$X_1 X_3 (2F - I)_{24}, \quad Z_1 Z_3 (2F - I)_{24}, \quad I_1 I_3 (2F - I)_{24}.$$
Example: the Heisenberg model

So, using Heisenberg interactions alone, we can implement an arbitrary (logical) Hamiltonian of the form

\[
H = \sum_{k=1}^{n} (\alpha_k X_k + \beta_k Z_k) I_{k'} + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j) (2F - I)_{i'j'},
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where we identify the \(i'\)th logical qubit pair with indices \((i, i')\).
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- We would like to remove the \((2F - I)\) operators.
- To do this, we force the primed qubits to be in some state by very strong \( F_{i'j'} \) interactions: we add the (logical) term

\[ G = \Delta \sum_{i<j} w_{ij} F_{i'j'}, \]

where \( w_{ij} \) are some weights and \( \Delta \) is very large.
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where \(w_{ij}\) are some weights and \(\Delta\) is very large.
- We can do this by making \(I_i I_j (2F - I)_{i'j'}\) as on last slide.
Example: the Heisenberg model

If the ground state $|\psi\rangle$ of $G$ is non-degenerate, the primed qubits will all be effectively projected onto the ground state, and $H$ will become (up to a small additive error)

$$\tilde{H} = \sum_{k=1}^{n} \alpha_k X_k + \beta_k Z_k + \sum_{i<j} (\gamma_{ij} X_i X_j + \delta_{ij} Z_i Z_j) \langle \psi | (2F - I)_{i'j'} | \psi \rangle.$$
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- So we need to find a $G$ such that the ground state is non-degenerate and $\langle \psi | (2F - I)_{i'j'} | \psi \rangle \neq 0$ for all $i, j$ (and also these quantities should be easily computable).
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- Luckily for us, the Lieb-Mattis model [Lieb and Mattis ‘62] has precisely the properties we need.
The Lieb-Mattis model

The Lieb-Mattis model describes Hamiltonians of the form

$$H_{LM} = \sum_{i \in A, j \in B} X_i X_j + Y_i Y_j + Z_i Z_j,$$

where $A$ and $B$ are disjoint subsets of qubits.
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where $A$ and $B$ are disjoint subsets of qubits.

Claim [Lieb and Mattis '62, ...]

If $|A| = |B| = n$, the ground state $|\phi\rangle$ of $H_{LM}$ is **unique**. For $i$ and $j$ such that $i, j \in A$ or $i, j \in B$, $\langle \phi | F_{ij} | \phi \rangle = 1$. Otherwise, $\langle \phi | F_{ij} | \phi \rangle = -2/n$. 

Using this claim, we can effectively implement any Hamiltonian of the form

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which suffices for QMA-completeness [Biamonte and Love '08].
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The other QMA-complete cases

We’ve dealt with the Heisenberg model... what about everything else?

- Our normal form drastically reduces the number of interactions we have to consider to a few special cases.
- The XY model $\mathcal{S} = \{XX + YY\}$ uses similar techniques to the Heisenberg model, but the gadgets are a bit simpler.
- For $\mathcal{S} = \{XX + \alpha YY + \beta ZZ\}$, we can reduce from the XY model.
- For interactions with 1-local terms, using gadgets we can effectively delete the 1-local parts.
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Finding and verifying each of the gadgets required was somewhat painful and required the use of a computer algebra package.
Conclusions and open problems

We have (almost) completely characterised the complexity of 2-local qubit Hamiltonians.

Despite this, our work is only just beginning...
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Despite this, our work is only just beginning. . .

- What about $k$-qubit interactions for $k > 2$? We only resolved this case for $S$-Hamiltonian with local terms.
Conclusions and open problems

We have (almost) completely characterised the complexity of 2-local qubit Hamiltonians.

Despite this, our work is only just beginning. . .

- What about $k$-qubit interactions for $k > 2$? We only resolved this case for S-HAMILTONIAN WITH LOCAL TERMS.

- What about local dimension $d > 2$? Classically, the complexity of $d$-ary CSPs is still unresolved.
More open problems

- What about restrictions on the interaction pattern or weights? e.g. the antiferromagnetic Heisenberg model etc.

- See very recent independent work proving QMA-hardness for $S = \{XX + YY, Z\}$ when weights of $XX + YY$ terms are positive and weights of $Z$ terms are negative [Childs, Gosset and Webb ’13].

- What about quantum $k$-SAT?

- Finally, what is the complexity of TIM? Our intuition: at least MA-hard.
Thanks!

arXiv:1311.3161
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To finish off the 2-local special case of $S$-HAMILTONIAN WITH LOCAL TERMS:

- If the 2-local part of any interaction in $S$ is locally equivalent to $XX + \beta YY + \gamma ZZ$ or $XZ - ZX$, we have QMA-completeness;
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- If the 2-local part of all the interactions is locally equivalent to $ZZ$, using local rotations we can show equivalence to the transverse Ising model;

- If neither of these is true, we must have one interaction equivalent to $XX$, another to $AA$ for some $A \neq X$ (exercise!).

- So we can make $XX + AA$, which suffices for QMA-completeness.
The $k$-local case for $k > 2$

We can generalise to $S$-HAMILTONIAN with local terms when $S$ contains $k$-qubit interactions, for any constant $k > 2$. 
**The $k$-local case for $k > 2$**

We can generalise to $S$-HAMILTONIAN WITH LOCAL TERMS when $S$ contains $k$-qubit interactions, for any constant $k > 2$.

- Basic idea: using local terms, produce effective $(k - 1)$-qubit interactions from $k$-qubit interactions, via the gadget

\[
\Delta |\psi\rangle \langle \psi| \quad I \otimes A + X \otimes B + Y \otimes C + Z \otimes D
\]

By letting $|\psi\rangle$ be the eigenvector of $X$, $Y$ or $Z$ with eigenvalue $\pm 1$, we can produce the effective interactions $A \pm B$, $A \pm C$ and $A \pm D$ (up to a small additive error). By adding/subtracting these matrices we can make each of \{A, B, C, D\}. So either $S$ is QMA-complete, or all 2-local "parts" of each interaction in $S$ are simultaneously diagonalisable by local unitaries. This case turns out to be in TIM.
The $k$-local case for $k > 2$

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So either $S$ is QMA-complete, or all 2-local "parts" of each interaction in $S$ are simultaneously diagonalisable by local unitaries. This case turns out to be in TIM.
The \( k \)-local case for \( k > 2 \)

We can generalise to \( S \)-Hamiltonian with local terms when \( S \) contains \( k \)-qubit interactions, for any constant \( k > 2 \).

- Basic idea: using local terms, produce effective \((k-1)\)-qubit interactions from \( k \)-qubit interactions, via the gadget

\[
\Delta |\psi\rangle \langle \psi| \quad I \otimes A + X \otimes B + Y \otimes C + Z \otimes D
\]

- By letting \( |\psi\rangle \) be the eigenvector of \( X \), \( Y \) or \( Z \) with eigenvalue \( \pm 1 \), we can produce the effective interactions \( A \pm B \), \( A \pm C \) and \( A \pm D \) (up to a small additive error).

- By adding/subtracting these matrices we can make each of \( \{A, B, C, D\} \).

- So either \( S \) is \textbf{QMA-complete}, or all 2-local “parts” of each interaction in \( S \) are simultaneously diagonalisable by local unitaries. This case turns out to be in \textbf{TIM}.
S-HAMILTONIAN: The list of lemmas

It suffices to prove QMA-completeness of the following cases:

1. \( \{XX + YY + ZZ\}\)-HAMILTONIAN;

2. \( \{XX + YY\}\)-HAMILTONIAN;

3. \( \{XZ - ZX\}\)-HAMILTONIAN;

4. \( \{XX + \beta YY + \gamma ZZ\}\)-HAMILTONIAN;

5. \( \{XX + \beta YY + \gamma ZZ + AI + IA\}\)-HAMILTONIAN;

6. \( \{XZ - ZX + AI - IA\}\)-HAMILTONIAN.

In the above, \( \beta, \gamma \) are real numbers such that at least one of \( \beta \) and \( \gamma \) is non-zero, and \( A \) is an arbitrary single-qubit Hermitian matrix.
**S-HAMILTONIAN: The list of lemmas**

We also need some reductions from cases which are not necessarily QMA-complete:

- \{ZZ, X, Z\}\text{-HAMILTONIAN} reduces to \{ZZ + AI + IA\}\text{-HAMILTONIAN};
- \{ZZ, X, Z\}\text{-HAMILTONIAN} reduces to \{ZZ, AI - IA\}\text{-HAMILTONIAN}.

In the above, \(A\) is any single-qubit Hermitian matrix which does not commute with \(Z\).

And the very final case to consider:

- Let \(S\) be a set of diagonal Hermitian matrices on at most 2 qubits. Then, if every matrix in \(S\) is \(1\)-local, \(S\)-HAMILTONIAN is in \(P\). Otherwise, \(S\)-HAMILTONIAN is \(NP\)-complete.
Example gadget for cases with 1-local terms

Let $H := XX + \beta YY + \gamma ZZ + AI + IA$, where $\beta$ or $\gamma$ is non-zero.

**Lemma**

{$H$}-Hamiltonian is QMA-complete.

The gadget used looks like:

- The ground state of $G := H_{ab} + H_{cd} - H_{ac} - H_{bd}$ is maximally entangled across the split $(a-c : d)$.
- So if we project $H_{de}$ onto this state, the effective interaction produced is $A$ on qubit $e$.
- This allows us to effectively delete the 1-local part of $H$. 

![Diagram](image-url)