Quantum walks on directed graphs

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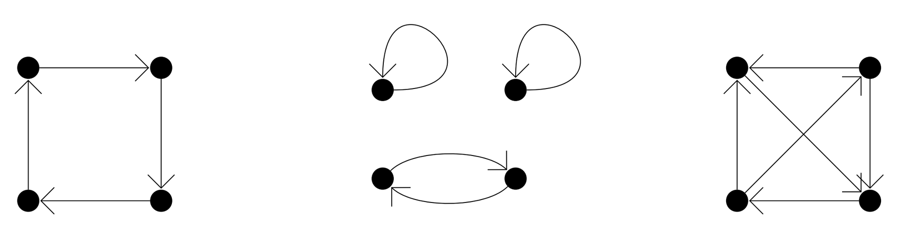
Introduction

- Quantum walks are a new model for quantum computation that have been used to produce novel quantum algorithms (e.g. [3]) on undirected graphs.
- We consider the definition of quantum walks on directed graphs.
- **Question:** Can quantum walks be meaningfully defined on all directed graphs?
- **Question:** Can quantum walks outperform classical algorithms operating on directed graphs?
- Continuous-time quantum walks [2] cannot be defined on directed graphs (as the graph's adjacency matrix must be Hermitian). So we only consider discrete-time quantum walks.

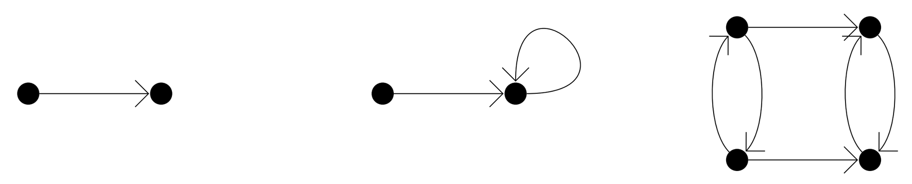
Reversible & irreversible graphs

A graph is a set of vertices and arcs. In an undirected graph, for each arc $a \to b$, there is a corresponding arc $b \to a$.

We say an arc $a \to b$ is reversible if there is a path from b to a. A graph whose arcs are all reversible is also called reversible; otherwise, it is called irreversible.



Some reversible graphs



Some irreversible graphs

In graph theoretic terms, every component of a reversible graph is strongly connected.

Examples of graph classification

- All undirected graphs are reversible.
- Any graph containing a source or sink is irreversible.
- All regular graphs are reversible.

Quantum walks

Use a generalised notion of quantum walks [1].

Definition: A discrete-time quantum walk is the repeated application of a unitary operator W. Each use of W is one step of the walk.

To define a quantum walk on a graph G, identify a finite set of ≥ 1 basis states $\{|v_i^1\rangle, |v_i^2\rangle, ...\}$ with each vertex v_i of the graph. A quantum walk can be implemented on G if there exists a W such that, for all i and j, $v_i \to v_j$ if and only if there exist k, l such that $\langle v_j^k|W|v_i^l\rangle \neq 0$.

This generalises the coined quantum walk where we use a "vertex & coin" Hilbert space $\mathcal{H}_v \otimes \mathcal{H}_c$. Identify a basis state in $\mathcal{H}_v$ with each vertex of the graph, and identify a basis state in $\mathcal{H}_c$ with each "coin toss" outcome. W is split into a coin toss and a shift: $W = S(I \otimes C)$.

Main result

Theorem: A discrete-time quantum walk can be defined on a finite graph G if and only if G is reversible.

Proof of necessity

We use the following lemma (from the Quantum Recurrence Theorem [4])

Lemma: For any vector $|a\rangle$ in a finite-dimensional Hilbert space, any unitary operator W, and any $\epsilon > 0$, there exists $n \geq 1$ such that $|\langle a|W^n|a\rangle| > 1 - \epsilon$.

Then show the following:

Lemma: For any vectors $|a\rangle$, $|b\rangle$ in a finite-dimensional Hilbert space, and for any unitary operator W, if $\langle b|W|a\rangle \neq 0$, then there exists $m \geq 0$ such that $\langle a|W^m|b\rangle \neq 0$.

Proof of sufficiency

Show that a coined quantum walk can be produced for any reversible graph G, because every arc in a reversible graph is included in at least one cycle.

To construct a coined quantum walk:

- Find a set S of cycles of G such that every arc in G is included in at least one cycle.
- Use a coin with $|S|$ different states, where each coin state is associated with a shift along a different cycle – a permutation of the vertices of G.

The Reachability problem

Problem: Given two vertices a, b in a graph G, is there a path from a to b?

- For undirected graphs this problem is in **L** [5].
- However, for directed graphs, this problem is **NL**-complete and thus expected to be harder.
- One classical solution is a random walk.

Question: Can we use a quantum walk algorithm to solve Reachability more quickly for directed graphs?

Answer: For reversible directed graphs, this problem reduces to undirected reachability. This implies that quantum walk algorithms might not be much help with Reachability.

Conclusions

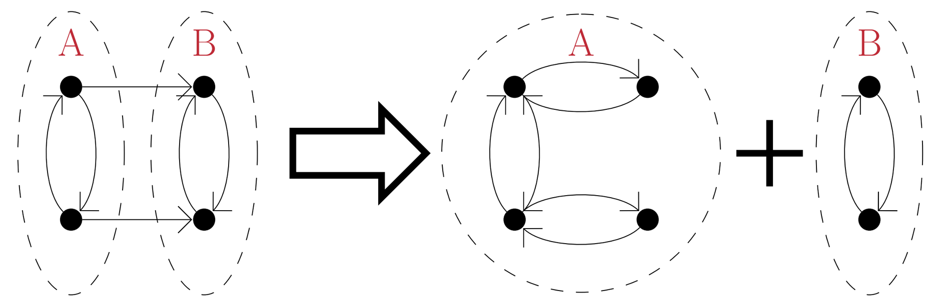
- Fully quantum walks that respect the structure of their underlying graph can only be defined on reversible graphs.
- All reversible graphs admit the definition of a coined quantum walk.
- We can produce a "partially quantum" walk on irreversible graphs, which maintains some coherence, using measurement.
- The Reachability problem is as easy for reversible graphs as it is for undirected graphs.
- Quantum walk algorithms for directed graphs may not be simple generalisations of classical random walk algorithms.
- Note that none of this considers the effect of modifying the structure of the graph (e.g. changing irreversible arcs to reversible ones).

Walks with measurement

- **Question:** How can we define a quantum walk on an irreversible graph G?
- **Answer:** By making use of the irreversible process of measurement.
- **Theorem:** A "partially quantum" walk using measurement can be defined on any directed graph.
- Define a reversible subgraph of a graph G to be a subgraph of G which, if considered as a graph itself, is reversible.

Construction of a quantum walk with measurement

1. Partition G into n reversible subgraphs, possibly connected to other subgraphs by irreversible arcs. For each irreversible arc leaving a subgraph, add an arc in the opposite direction.



Dividing an irreversible graph into reversible subgraphs

2. Define n standard coined quantum walks $\{W_i\}$: one walk on each reversible subgraph.
3. Define an incomplete measurement M that has n outcomes and projects onto the partition into subgraphs.

Execution of a walk with measurement

Repeat the following two steps:

1. Measure M to determine which reversible subgraph the walker is in.
2. If we see outcome i from the measurement, perform unitary walk operation W_i.

This approach has the advantage that coherence can be maintained when walking in a reversible subgraph, or when moving from one to another. However, it is not possible to maintain coherence across different subgraphs.

The more irreversible arcs in the graph, the more "classically" the quantum walk will behave.

References

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[2] E. Farhi, S. Gutmann. Quantum computation and decision trees. *Phys. Rev. A* 58, pp. 915-928, 1998, quant-ph/9706062

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[4] P. Bocchieri, A. Loinger. Quantum Recurrence Theorem. *Phys. Rev. 107, 337*, 1957

[5] O. Reingold, Undirected ST-Connectivity in Log-Space, ECCC 94

quant-ph/0504116