The quantum threat to cryptography

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20 October 2016
Quantum computers
Experimental progress

Important aspects of a quantum computer are:

- The number of **qubits** (quantum bits) it has;
- The number of **quantum gates** (elementary operations) it can execute, and the speed with which it does so;
- Whether it is **fault-tolerant** and **scalable**.
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Estimates for when this will be achieved vary, but only a pessimist would bet $1M on it taking $>20$ years...
One of the important applications of quantum computers is expected to be attacking cryptosystems that are designed to be secure against classical adversaries.

The rest of this talk:

1. Efficient quantum attacks on public-key cryptosystems;
2. General-purpose quantum algorithms and applications to cryptographic tasks.
Integer factorisation

**Problem**

Given an $n$-digit integer $N = p \times q$ for primes $p$ and $q$, determine $p$ and $q$. 

The best (classical!) algorithm we have for factorisation (the number field sieve) runs in time $\exp\left(O\left(\frac{n^{1/3}}{(\log n)^{2/3}}\right)\right)$.

The RSA cryptosystem is based around the hardness of this task. If we can factorise large integers efficiently, we can break RSA.

**Theorem** [Shor '97]

There is a quantum algorithm which finds the prime factors of an $n$-digit integer in time $O(n^3)$. 

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Shor’s algorithm: complexity comparison

Very roughly (ignoring constant factors!):

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Based on these figures, a 10,000-digit number could be factorised by:

- A quantum computer with a clock speed of 1MHz in 11 days.
- The fastest computer on the Top500 supercomputer list (~ $9.3 \times 10^{16}$ operations per second) in $\sim 3.4 \times 10^{16}$ years.

(see e.g. [Van Meter et al '05] for a more detailed comparison)
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But a cautionary note...

Figure 58: The relation between speed of a quantum computer and the size of number that can be factorised in 1 day

Pic: Nicharee Techatanerut, 2014
Hidden subgroup problems

Hidden subgroup problem (e.g. [Boneh and Lipton ‘95])

Let $G$ be a group. Given oracle access to a function $f : G \to X$ such that $f$ is **constant** on the cosets of some subgroup $H \leq G$, and **distinct** on each coset, identify $H$. 

Integer factorisation reduces to the case $G = \mathbb{Z}/M$ for some integer $M$. This is the problem of determining the period of a periodic function.

On a quantum computer, the HSP can be solved using $O(\log |G|)$ queries to $f$ for all groups $G$ [Ettinger et al. ‘04].

Classically, some groups require $\Omega(\sqrt{|G|})$ queries [Simon ‘97].
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![Periodic Function Example](image-url)
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## Hidden subgroup problems

The HSP is related to many other problems and cryptosystems:

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<td>Polynomial$^1$</td>
<td>Diffie-Hellman, DSA, …</td>
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<td>Elliptic curve d. log</td>
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<td>Polynomial$^2$</td>
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<tr>
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<td>Dihedral grp</td>
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|                                | Elliptic curve         | Polynomial$^1$ | 
|                                | $\mathbb{R}$           | Polynomial$^2$ | ECDH, ECDSA,  
|                                | $\mathbb{R}^m$         | Polynomial$^3$ | 
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| Shortest lattice vector        |                        | Exponential    | NTRU, Ajtai-Dwork,  
| Graph isomorphism              |                        |                | —                   |

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A significant amount of other work on the HSP has resolved its complexity for many other groups.
Other cryptosystems?

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- **May 2006**: first conference on post-quantum crypto held.
- **2014-2016**: post-quantum crypto companies emerge: e.g. Post-Quantum, ISARA, …?
- **Aug 2015**: NSA states that “we anticipate a need to shift to quantum-resistant cryptography in the near future”
- **July 2016**: Google announces that a candidate post-quantum cryptosystem (“New Hope”) has been implemented as an experiment in Chrome.
“Post-quantum” cryptosystems

Some examples of cryptosystems which have thus far resisted quantum attack:

- The McEliece cryptosystem, which is (roughly) based on the hardness of finding transformations between equivalent linear codes.

There can be no efficient quantum attack on this cryptosystem based on simple Fourier sampling (the key ingredient in Shor’s algorithm) [Dinh et al. ‘10].

Lattice-based cryptosystems dependent on the hardness of solving closest/shortest vector problems in lattices. No polynomial-time quantum algorithm for these problems has been found for general lattices (but there is an efficient algorithm for lattices with more structure [e.g. Campbell et al. ‘14, Biasse and Song ‘15]).
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Grover’s algorithm

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\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
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- We want to find an $x$ such that $f(x) = 1$.

- On a classical computer, this task could require $2^n$ queries to $f$ in the worst case. But on a quantum computer, Grover’s algorithm [Grover ‘97] can solve the problem with $O(\sqrt{2^n})$ queries to $f$ (and bounded failure probability).
Applications of Grover’s algorithm

Grover’s algorithm gives a speedup over naïve algorithms for any decision problem in the complexity class $\text{NP}$, i.e. where we can verify the solution efficiently.
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For example, in the Circuit SAT problem we would like to find an input to a circuit on $n$ bits such that the output is 1:

- AND
- OR
- NOT
- AND

1

1

1

0

1
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![Circuit Diagram](image)

- Grover’s algorithm improves the runtime from \( O(2^n) \) to \( O(2^{n/2} \text{poly}(n)) \): applications to design automation, circuit equivalence, model checking, \ldots
Quadratic speedup

Is a quadratic speedup significant?
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A concrete example: Circuit SAT with different clock speeds.

<table>
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<th>Input bits</th>
<th>Classical 1MHz</th>
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<tr>
<td>30</td>
<td>18s</td>
<td>1s</td>
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<td>3s</td>
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<td>40</td>
<td>13d</td>
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<tr>
<td>50</td>
<td>36y</td>
<td>13d</td>
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Speeds listed are approximate, effective speeds (i.e. number of circuit evaluations per second) after overhead for fault-tolerance.
Cryptographic applications of Grover’s algorithm

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In all these cases, one need only increase the key length by a constant factor to achieve the same level of security as was the case classically.
Other notes on Grover’s algorithm

Grover’s algorithm is not parallelisable in the following sense:

- Imagine we have $K$ quantum or classical computers solving a search problem in a space of size $N$.

- Classical complexity: $O(N/K)$ per computer $\Rightarrow$ total effort $O(N)$.

- Quantum complexity: $O(\sqrt{N/K})$ per computer $\Rightarrow$ total effort $O(\sqrt{NK})$. 
Finding hash function collisions

Quantum computers can also be used to find collisions in hash functions etc. more efficiently than classically:

- [Brassard, Høyer and Tapp ’98] gave a quantum algorithm finding a collision in an 2-to-1 function $f : [N] \rightarrow X$ using the function $O(N^{1/3})$ times.
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Finding a collision without the 2→1 promise can be done with \( O(N^{2/3}) \) function evaluations [Ambainis ’04], and this is tight.
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Quantum speedup of backtracking algorithms

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Can be applied e.g. to speed up enumeration attacks on lattice-based cryptosystems [del Pino et al. ’16, Alkim et al. ’16].
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See the Quantum Algorithm Zoo for over 320 papers on quantum algorithms: http://math.nist.gov/quantum/zoo/

Quantum algorithms: an overview, AM, npj Quantum Information 2, 2016
www.nature.com/articles/npjqi201523