A quantum analogue of Fourier analysis on the boolean cube

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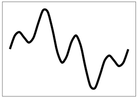
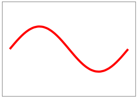
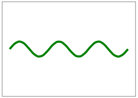
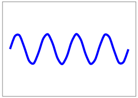
Talk based on joint work with Tobias Osborne

University of Bristol

QICS

Fourier analysis

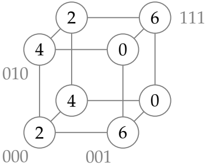
...traditionally looks like this:

 $= 3\times$  $-$  $+2\times$ 

- Given some function $f : \mathbb{R} \to \mathbb{R}$...
- ...we expand it in terms of trigonometric functions $\sin(kx)$, $\cos(kx)$...
- ...in an attempt to understand the structure of f.

Fourier analysis

In computer science, it's natural to consider functions on the set of n-bit strings – also known as the boolean cube $\{0,1\}^n$:



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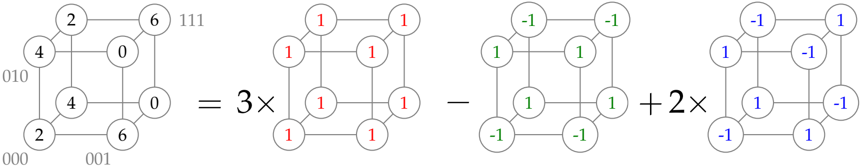

- Given some function $f : \{0,1\}^n \to \mathbb{R}$...
- ...we expand it in terms of parity functions...
- ...in an attempt to understand the structure of f.

This talk

- The classical theory of Fourier analysis on the boolean cube
- A quantum generalisation
- Application: Testing for Pauli operators
- The qubit depolarising channel
- Application: Spectra of k-local operators

Fourier analysis on the boolean cube

We expand functions $f : \{0,1\}^n \to \mathbb{R}$ in terms of the parity functions

$$\chi_S(x) = (-1)^{\sum_{i \in S} x_i},$$

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Any $f : \{0,1\}^n \to \mathbb{R}$ has the expansion

$$f = \sum_{S \subseteq \{1,\ldots,n\}} \hat{f}_S \, \chi_S$$

for some $\{\hat{f}_S\}$ – the Fourier coefficients of f.

Applications of Fourier analysis on the boolean cube

This approach has led to new results in many areas of classical computer science, including:

- Probabilistically checkable proofs [Håstad '01; Dinur '07; . . .]
- Decision tree complexity [Nisan & Szegedy '94]
- Influence of voters and fairness of elections [Kahn, Kalai, Linial '88; Kalai '02]
- Computational learning theory [Goldreich & Levin '89; Kushilevitz & Mansour '91; . . .]
- Property testing [Bellare et al '95; Matulef et al '09; . . .]

Property testing

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- We want to output whether f has some property P, or is "far" from having property P, using a constant number of queries.
- Sample problem: Determine whether f is linear, or "far" from linear: i.e. differs from all linear functions in a constant fraction of places.

Applications in quantum computation

There have also been some recent applications of Fourier analysis to quantum computer science.

- Quantum algorithms for computational learning
  [Bshouty & Jackson '95; Atici & Servedio '07]
- Quantum communication complexity
  [Klauck '01; Gavinsky et al '07]
- Lower bounds on quantum locally decodable codes
  [Ben-Aroya, Regev, de Wolf '08]
- Quantum algorithms with exponential speed-ups
  [Roetteler '08; AM '08]

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Our generalisation: instead of decomposing functions $\{0,1\}^n \to \mathbb{C}$, we decompose linear operators on the space of n qubits.

“Fourier analysis” for qubits

It turns out that a natural analogue of the characters of $\mathbb{Z}_2$ are the Pauli matrices:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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$\chi_{\mathbf{s}} \equiv \sigma^{s_1} \otimes \sigma^{s_2} \otimes \cdots \otimes \sigma^{s_n}$, where $s_j \in \{0, 1, 2, 3\}$.

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Any n qubit linear operator f has an expansion

$$f = \sum_{\mathbf{s} \in \{0,1,2,3\}^n} \hat{f}_{\mathbf{s}}\, \chi_{\mathbf{s}}.$$

for some $\{\hat{f}_{\mathbf{s}}\}$ – the Pauli coefficients of f. This is our analogue of the Fourier expansion of a function $f : \{0, 1\}^n \to \mathbb{C}$.

Norms and closeness

Some definitions we'll need later:

- The (normalised) Schatten p-norm: for any d-dimensional operator M,

$$\|M\|_p \equiv \left(\frac{1}{d} \sum_{j=1}^{d} \sigma_j^p \right)^{\frac{1}{p}},$$

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- Closeness: Let U and V be two linear operators. Then we say that f and g are ϵ-close if $\|U - V\|_2^2 \leqslant 4\epsilon$.

Quantum property testing

Consider the following representative example:

Pauli testing

Given oracle access to an unknown unitary operator U on n qubits, determine whether U is a Pauli operator $\chi_{\mathbf{s}}$ for some $\mathbf{s}$.

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We give a test (the quantum Pauli test) that has the following property.

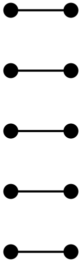
Proposition

Suppose that a unitary operator U passes the quantum Pauli test with probability $1 - \epsilon$. Then U is ϵ-close to a Pauli operator (with phase) $e^{i\phi}\chi_{\mathbf{s}}$.

The test uses 2 queries (best known classical test uses 3).

Quantum Pauli testing algorithm (sketch)

1. Apply U to the first halves of n Bell pairs $|\Phi\rangle^{\otimes n}$, resulting in a quantum state $|u\rangle = U \otimes \mathbb{I}|\Phi\rangle^{\otimes n}$.



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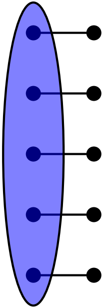

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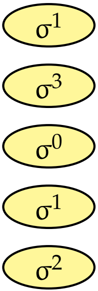

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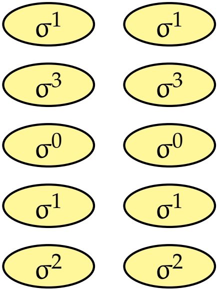

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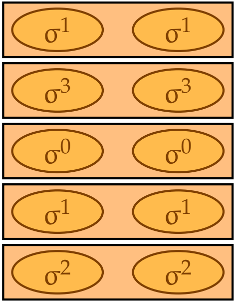

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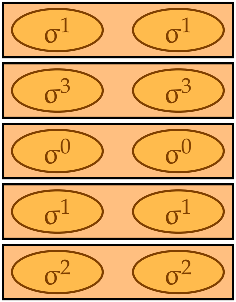
5. Accept if all measurements say "yes".



It turns out that for the Pauli test $\Pr[\text{test accepts}] = \sum_{\mathbf{s}} |\hat{U}_{\mathbf{s}}|^4$, which implies the proposition by Parseval's equality.

The Pauli operators and depolarising noise

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- Let $\mathcal{D}_\epsilon$ be the qubit depolarising channel with noise rate $1-\epsilon$, i.e.

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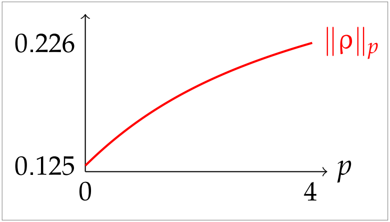
We are interested in the smoothing effect of this channel.

The qubit depolarising channel and p-norms

p-norms of a random quantum state ρ increase with p:



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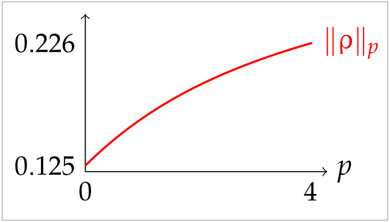
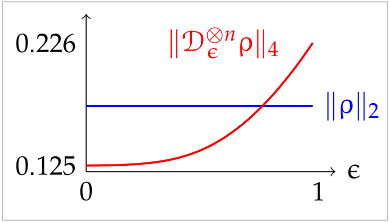


Applying depolarising noise smooths ρ by reducing its higher norms:



Quantum hypercontractivity

Proposition

Let H be a Hermitian operator on n qubits and assume that $1 \leqslant p \leqslant 2 \leqslant q \leqslant \infty$. Then, provided that $\epsilon \leqslant \sqrt{\frac{p-1}{q-1}}$, we have

$$\|\mathcal{D}_\epsilon^{\otimes n}(H)\|_q \leqslant \|H\|_p.$$

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- This is a quantum generalisation of a hypercontractive inequality of Bonami, Gross and Beckner for functions $f : \{0,1\}^n \to \mathbb{R}$, which is an essential component in many results in classical analysis of boolean functions.
- The quantum proof isn't a simple generalisation of the classical proof, but would be if the maximum output $p \to q$ norm were multiplicative!

Application: Spectra of k-local operators

A Hamiltonian H on n qubits is said to be k-local if it has a decomposition

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Our results show that the spectra of k-local operators are "smooth". In particular:

- For any $q \geqslant 2$, $\|H\|_q \leqslant (q-1)^{k/2}\|H\|_2$
- $\text{rank}(H) \geqslant 2^{n-O(k)}$ (a quantum Schwartz-Zippel lemma)
- ...

Conclusions

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We still have many open conjectures... such as:

- Conjecture: There exists an efficient quantum property tester for dictators.
- Conjecture: Every traceless operator $U^2 = \mathbb{I}$ has an influential qubit: there is a j such that $\| \operatorname{tr}_j U \otimes \mathbb{I}/2 - U\|_2^2 = \Omega((\log n)/n)$.
- ...

The end

Further reading:

- "Quantum boolean functions", AM & Tobias Osborne, arXiv:0810.2435.
- "Learning and testing algorithms for the Clifford group", Richard Low, arXiv:0907.2833.
- Survey paper by Ronald de Wolf: http://theoryofcomputing.org/articles/gs001/gs001.pdf
- Lecture course by Ryan O'Donnell: http://www.cs.cmu.edu/~odonnell/boolean-analysis/

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Thanks for your time!

Application: A quantum FKN theorem

- The classical FKN (Friedgut-Kalai-Naor) theorem: Let f be a boolean function. Then, if $\sum_{|S|>1} \hat{f}_S^2 < \epsilon$, f is $O(\epsilon)$-close to depending on 1 variable (being a dictator).

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Quantum FKN theorem

Let U be a unitary operator on n qubits with eigenvalues ± 1. If

$$\sum_{|\mathbf{s}|>1} \hat{U}_{\mathbf{s}}^2 < \epsilon,$$

then there is a constant K such that U is $K\epsilon$-close to being a dictator (acting non-trivially on only 1 qubit) or the identity.

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We give a quantum algorithm that outputs the large Pauli coefficients of U. If U is almost completely determined by these, this is sufficient to approximately learn U.

Computational learning of unitary operators

"Quantum Goldreich-Levin" algorithm

Given oracle access to a unitary U, and given $\gamma, \delta > 0$, there is a poly $\left(n, \frac{1}{\gamma}\right) \log\left(\frac{1}{\delta}\right)$-time algorithm which outputs a list $L = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_m\}$ such that with prob. $1 - \delta$: (1) if $|\hat{U}_{\mathbf{s}}| \geqslant \gamma$, then $\mathbf{s} \in L$; and (2) if $\mathbf{s} \in L$, $|\hat{U}_{\mathbf{s}}| \geqslant \gamma/2$.

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Example: learning dynamics of a 1D spin chain. Informally:

Theorem

Let H be a Hamiltonian corresponding to an n-site spin chain, and let $t = O(\log n)$. Then we can approximately learn the operators $\sigma_j^s(t) \equiv e^{-itH} \sigma_j^s e^{itH}$ with $\mathrm{poly}(n)$ uses of e^{itH}.

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Example: learning dynamics of a 1D spin chain. Informally:

Theorem

Let H be a Hamiltonian corresponding to an n-site spin chain, and let $t = O(\log n)$. Then we can approximately learn the operators $\sigma_j^s(t) \equiv e^{-itH}\sigma_j^s e^{itH}$ with poly(n) uses of e^{itH}.

So we can predict the outcome of measuring σ^s on site j after a short time, on average over all input states.