All-pairs shortest paths

Ashley Montanaro
ashley@cs.bris.ac.uk
Department of Computer Science, University of Bristol
Bristol, UK
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Today: algorithms for general graphs with better runtimes than this.

- The Floyd-Warshall algorithm: time $O(V^3)$.
- Johnson's algorithm: time $O(VE + V^2 \log V)$.

Assume for simplicity that the input graph has no negative-weight cycles.

We have seen two different ways of determining the shortest path from a vertex $s$ to all other vertices.

- What if we want to determine the shortest paths between all pairs of vertices?
- For example, we might want to store these paths in a database for efficient access later.

We could use Dijkstra (if the edge weights are non-negative) or Bellman-Ford, with each vertex in turn as the source, which would achieve complexity $O(VE + V^2 \log V)$ and $O(V^2E)$ respectively.

Can we do better?

A dynamic-programming recurrence

Let $p$ be a shortest (i.e. minimum-weight) path from $i$ to $j$ with all intermediate vertices in the set $\{1, \ldots, k\}$. Then observe that:

- If $k$ is not an intermediate vertex of $p$, then $p$ is also a minimum-weight path with all intermediate vertices in the set $\{1, \ldots, k-1\}$.
- If $k$ is an intermediate vertex of $p$, then we decompose $p$ into a path $p_1$ between $i$ and $k$, and a path $p_2$ between $k$ and $j$.

By the triangle inequality, $p_1$ is a shortest path from $i$ to $k$. Further, it does not include $k$ (as otherwise it would contain a cycle).

The same reasoning shows that $p_2$ is a shortest path from $k$ to $j$.

We therefore have the following recurrence for $d^{(k)}_{ij}$:

$$d^{(k)}_{ij} = \begin{cases} W_{ij} & \text{if } k = 0 \\ \min \{ d^{(k-1)}_{ij}, d^{(k-1)}_{ik} + d^{(k-1)}_{kj} \} & \text{if } k \geq 1. \end{cases}$$
The Floyd-Warshall algorithm

Based on the above recurrence, we can give the following bottom-up algorithm for computing $d^{(n)}_i$ for all pairs $i, j$.

**FloydWarshall(W)**

1. $d^{(0)} \leftarrow W$
2. for $k = 1$ to $n$
3.   for $i = 1$ to $n$
4.     for $j = 1$ to $n$
5.       $d^{(k)}_i \leftarrow \min(d^{(k-1)}_i, d^{(k-1)}_{ik} + d^{(k-1)}_{kj})$
6. return $d^{(n)}$.

- The time complexity is clearly $O(n^3)$ and the algorithm is very simple.
- Correctness follows from the argument on the previous slide.

**Example**

Consider the following graph and its corresponding adjacency matrix:

\[
\begin{pmatrix}
0 & 1 & \infty & \infty \\
\infty & 0 & 1 & \infty \\
2 & 4 & 0 & 0 \\
-1 & \infty & \infty & 0
\end{pmatrix}
\]

$\begin{pmatrix}
0 & 1 & 2 \\
2 & 3 & 0 & 0 \\
-1 & 0 & 1 & 0
\end{pmatrix}$

Constructing the shortest paths

- We would like to construct a predecessor matrix $\Pi$ such that $\Pi_{ij}$ is the predecessor vertex of $j$ in a shortest path from $i$ to $j$.
- We can do this in a similar way to computing the distance matrix. We define a sequence of matrices $\Pi^{(0)}, \ldots, \Pi^{(n)}$ such that $\Pi^{(k)}_{ij}$ is the predecessor of $j$ in a shortest path from $i$ to $j$ only using vertices in the set $\{1, \ldots, k\}$.
- Then, for $k = 0$,

  $\Pi^{(0)}_{ij} = \begin{cases} 
  \text{nil} & \text{if } i = j \text{ or } W_{ij} = \infty \\
  i & \text{if } i \neq j \text{ and } W_{ij} \neq \infty.
  \end{cases}$

- For $k \geq 1$, we have essentially the same recurrence as for $d^{(k)}$.

  Formally,

  $\Pi^{(k)}_{ij} = \begin{cases} 
  \Pi^{(k-1)}_{ij} & \text{if } d^{(k-1)}_{ij} = \infty \\
  \Pi^{(k-1)}_{ik} & \text{if } d^{(k-1)}_{ij} - d^{(k-1)}_{ik} \leq d^{(k-1)}_{ik} + d^{(k-1)}_{kj} \\
  \Pi^{(k-1)}_{kj} & \text{otherwise}.
  \end{cases}$

\[
\begin{pmatrix}
0 & 1 & 2 \\
2 & 3 & 0 & 0 \\
-1 & 0 & 1 & 0
\end{pmatrix}
\]
### The Floyd-Warshall algorithm with predecessors

**FloydWarshall(W)**

1. \(d^{(0)} \leftarrow W\)
2. for \(k = 1\) to \(n\)
   3. for \(i = 1\) to \(n\)
      4. for \(j = 1\) to \(n\)
         5. if \(d^{(k-1)}_{ij} \leq d^{(k-1)}_k + d^{(k-1)}_{kj}\)
            6. \(d^{(k)}_{ij} \leftarrow d^{(k-1)}_{ij}\)
            7. \(\Pi^{(k)}_{ij} \leftarrow \Pi^{(k-1)}_{ij}\)
         8. else
            9. \(d^{(k)}_{ij} \leftarrow d^{(k-1)}_k + d^{(k-1)}_{kj}\)
            10. \(\Pi^{(k)}_{ij} \leftarrow \Pi^{(k-1)}_{kj}\)
3. return \(d^{(n)}\).

### Johnson’s algorithm

- For sparse graphs with non-negative weight edges, running Dijkstra with each vertex in turn as the source is more efficient than the Floyd-Warshall algorithm.
- Johnson’s algorithm uses Dijkstra’s algorithm to solve the all-pairs shortest paths problem for graphs which may have negative-weight edges. It is based around the idea of first reweighting \(G\) so that all the weights are non-negative, then using Dijkstra.
- For sparse graphs, its complexity \(O(VE + V^2 \log V)\) (the same as Dijkstra) is faster than the Floyd-Warshall algorithm.
- We assume that we are given \(G\) as an adjacency list, and have access to a weight function \(w(u,v)\) which tells us the weight of the edge \(u \rightarrow v\).

### Claim

For any edge \(u \rightarrow v\), define

\[
\hat{w}(u,v) := w(u,v) + h(u) - h(v),
\]

where \(h\) is an arbitrary function mapping vertices to real numbers. Then any path \(p = v_0, \ldots, v_k\) is a shortest path from \(v_0\) to \(v_k\) with respect to the weight function \(\hat{w}\) if and only if it is a shortest path with respect to the weight function \(w\).

### Proof

The total weights of \(p\) under \(\hat{w}\) and \(w\) are closely related:

\[
\sum_{i=1}^{k} \hat{w}(v_{i-1}, v_i) = \sum_{i=1}^{k} w(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i) = h(v_0) - h(v_k) + \sum_{j=1}^{k} w(v_{j-1}, v_j) \ldots
\]
Negative-weight cycles

Claim

A graph has a negative-weight cycle under weight function \( \hat{w} \) if and only if it has one under weight function \( w \).

Proof

- Let \( c = v_0, \ldots, v_k \), where \( v_0 = v_k \), be any cycle.
- As \( v_0 = v_k \), \( h(v_0) = h(v_k) \), so the weight of \( c \) under \( \hat{w} \) is the same as its weight under \( w \).
- So \( c \) is negative-weight under \( \hat{w} \) if and only if it is negative-weight under \( w \).

\[ \square \]

Reweighting

- Given a graph \( G \), to define our new weight function, we add a new vertex \( s \) which has an edge of weight 0 to all other vertices in \( G \).
- This cannot create a new negative-weight cycle if there was not one there already.
- We then define \( h(v) = \delta(s, v) \) for all vertices \( v \) in \( G \).
- Now observe that \( \delta(s, v) \leq \delta(s, u) + w(u, v) \) for all edges \( u \rightarrow v \) by the triangle inequality, so \( h(v) - h(u) \leq w(u, v) \).
- So, if we reweight according to the function \( h \),
  \[ \hat{w}(u, v) = w(u, v) + h(u) - h(v) \geq 0 \]
  for all edges \( u \rightarrow v \).
- Then, if \( \hat{\delta}(u, v) \) is the weight of a shortest path from \( u \) to \( v \) with weight function \( \hat{w} \),
  \[ \hat{\delta}(u, v) = \delta(u, v) + h(v) - h(u) \]

Example

Imagine we want to reweight the following graph:

- Using Bellman-Ford, we compute \( h(A) = -2, \ h(B) = -1, \ h(C) = 0, \ h(D) = -1 \).

Example

Reweighting according to \( h \) gives the following graph:

- For each pair of vertices \( u, v \), \( \hat{\delta}(u, v) = \delta(u, v) + h(v) - h(u) \).
- For example, \( \hat{\delta}(C, A) = 0 - 2 - 0 = -2 \) as expected.
Johnson’s algorithm

From the above discussion, we can write down the following algorithm.

Johnson\((G)\)

1. form a new graph \(G'\) by adding \(s\) to \(G\), as defined above
2. compute \(\delta(s, v)\) for all \(v \in G\) using BellmanFord
3. for each edge \(u \to v\) in \(G\)
   \[
   \hat{w}(u, v) \leftarrow w(u, v) + \delta(s, u) - \delta(s, v)
   \]
4. for each vertex \(u \in G\)
   \[
   \hat{\delta}(u, v) \leftarrow \hat{\delta}(u, v) + \delta(s, v) - \delta(s, u)
   \]
5. for each vertex \(v \in G\)
6. compute \(d_{uv}\) for all \(v\) using Dijkstra
7. for each vertex \(v \in G\)
   \[
   d_{uv} \leftarrow \hat{\delta}(u, v) + \delta(s, v) - \delta(s, u)
   \]
9. return \(d\)

Summary of all-pairs shortest paths algorithms

We have now seen two different algorithms for this problem.

- Both algorithms work for graphs which may have negative-weight edges.
- The Floyd-Warshall algorithm runs in time \(O(V^3)\) and is based on ideas from dynamic programming.
- Johnson’s algorithm is based on reweighting edges in the graph and running Dijkstra’s algorithm.
- The runtime of Johnson’s algorithm is dominated by the complexity of running Dijkstra’s algorithm once for each vertex, which is \(O(VE + V^2 \log V)\) if implemented using a Fibonacci heap, and \(O(VE \log V)\) if implemented using a binary heap.
- This can be significantly smaller than the runtime of the Floyd-Warshall algorithm if the input graph is sparse.

Shortest path algorithms: the summary

To compute single-source shortest paths in a directed graph \(G\) which is...

- unweighted: use breadth-first search in time \(O(V + E)\);
- weighted with non-negative weights: use Dijkstra’s algorithm in time \(O(E + V \log V)\);
- weighted with negative weights: use Bellman-Ford in time \(O(VE)\).

To compute all-pairs shortest paths in a directed graph \(G\) which is...

- unweighted: use breadth-first search from each vertex in time \(O(VE + V^2)\);
- weighted with non-negative weights: use Dijkstra’s algorithm from each vertex in time \(O(VE + V^2 \log V)\);
- weighted with negative weights: use Johnson’s algorithm in time \(O(VE + V^2 \log V)\).

Further Reading

- Introduction to Algorithms
  Chapter 25 – All-Pairs Shortest Paths

- Algorithms lecture notes, University of Illinois
  Jeff Erickson
  http://www.cs.uiuc.edu/~jeffe/teaching/algorithms/
  Lecture 20 – All-pairs shortest paths
Biographical notes

The Floyd-Warshall algorithm was invented independently by Floyd and Warshall (and also Bernard Roy).

Robert W. Floyd (1936–2001)

- American computer scientist who did major work on compilers and initiated the field of programming language semantics.
- He completed his first degree (in liberal arts) at the age of 17 and won the Turing Award in 1978.
- Had his middle name legally changed to “W”.


- Another American computer scientist whose other work included operating systems and compiler design.
- Supposedly he and a colleague bet a bottle of rum on who could first prove correctness of his algorithm.
- Warshall found his proof overnight and won the bet (and the rum).

Donald B. Johnson (d. 1994)

- Yet another American computer scientist. Founded the computer science department at Dartmouth College and invented the $d$-ary heap.