Bloom filters

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19 November 2013
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These are all the operations we care about: that is, instead of supporting Insert, Delete, Find and Successor operations, we will just want to support Insert and Member.

The data structure maintains a subset $S \subseteq U$ of keys. The operation $\text{Member}(k)$ should just return whether or not the supplied key $k$ is contained within $S$. 
Bloom filters are a randomised data structure which achieve this goal. However, they have some important caveats:

- Bloom filters do not support deletion; they only support Insert and Member.

   - That is, when we query the Bloom filter with some key \( k \), if \( k \notin S \) there is some small chance (say 1%) that the answer is "yes" when it should be "no". On the other hand, if \( k \in S \) the answer is always "yes".

   - This is reasonable for applications like a web cache:
     - If we incorrectly think that a page is in the cache, this is not a disaster: we check the cache first, find it is not there, and download it directly.
     - However, if we incorrectly decide that a page is not in the cache, this is undesirable because we download the page unnecessarily.
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**Introduction**

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- However, if we incorrectly decide that a page is not in the cache, this is undesirable because we download the page unnecessarily.
Example

The following sequence of operations illustrates what can happen using a Bloom filter.

<table>
<thead>
<tr>
<th>Operation</th>
<th>Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insert(<a href="http://www.bbc.co.uk">www.bbc.co.uk</a>)</td>
<td></td>
</tr>
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The last “Yes” is an example of a false positive.
A naïve approach

- The simplest thing we could do to implement the web cache is to maintain a string $B$ of $U$ bits in an array, where bit $B[k]$ is set to 0 or 1 depending on whether $k \in S$. 

For example, if the universe is the integers between 1 and 10, after inserting 3, 6 and 8 we have:

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0 0 1 0 0 1 0 1 0 0
```

- If we would like the storage space used not to depend on $U$, we will need to compress this string somehow.
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One way to do this is by hashing. We maintain an $m$-bit string $B$ in our structure, for some $m$ to be determined. Assume we have access to a hash function $h$ which maps each key $k$ to an integer $h(k)$ between 1 and $m$. Our structure will set bit number $h(k)$ of $B$ to 1 when key $k$ is inserted. Then, to determine whether $k \in S$, we just check whether the bit of $B$ at position $h(k)$ is equal to 1.
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Example

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Start

0 0 0
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\[
\begin{array}{ccc}
\text{Start} & & \\
& 0 & 0 & 0 \\
\text{Insert(www.bbc.co.uk)} & & \\
& 0 & 1 & 0 \\
\end{array}
\]
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Start

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
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Insert(www.bbc.co.uk)

<p>| | | |</p>
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<th></th>
<th></th>
</tr>
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<tr>
<td>0</td>
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<td>0</td>
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Insert(facebook.com)

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Start

Insert(www.bbc.co.uk)

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Member(cs.bristol.ac.uk)

returns Yes
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- If we call Member(\( k \)) for some \( k \notin S \), if \( h(k) = h(k') \) for some \( k' \in S \), we will incorrectly output “yes”.

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- For each key $k$, the value of $h(k)$ is uniformly random: that is, the probability that $h(k) = j$ is equal to $1/m$ for all $j$ between 1 and $m$. 
What is the probability of a collision?

Assume we have already inserted $n$ keys into the structure and we would like to check whether some other key $k \notin S$ is contained in $S$ (so the output should be "no").

The bit-string $B$ contains at most $n$ 1's, and the value $h(k)$ is uniformly random; so the probability that $B[h(k)] = 1$ is at most $n/m$.

So the probability that we incorrectly output "yes" for this key is at most $n/m$, and we never incorrectly output "no" for any key.

So it suffices (for example) to take $m = 100n$ to achieve a failure probability of at most 1%. Note that $m$ does not depend on the universe size $U$.
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- We will choose the parameters $m$ and $r$ later.
Inserting into a Bloom filter

To insert into a Bloom filter, we use the following simple procedure.

**Insert**($k$)

1. for $i ← 1$ to $r$
2. $B[h_i(k)] ← 1$
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\[\text{Insert}(k)\]

1. for \( i \leftarrow 1 \) to \( r \)
2. \( B[h_i(k)] \leftarrow 1 \)

To check membership, we just check the bits of \( B \) that should be set to 1.

\[\text{Member}(k)\]

1. for \( i \leftarrow 1 \) to \( r \)
2. if \( B[h_i(k)] = 0 \)
3. return false
4. return true
Example

Imagine $m = 4$, $r = 2$, and we randomly pick the following hash functions:

- $h_1(\text{www.bbc.co.uk}) = 2$, $h_1(\text{facebook.com}) = 3$, $h_1(\text{cs.bristol.ac.uk}) = 3$.
- $h_2(\text{www.bbc.co.uk}) = 1$, $h_2(\text{facebook.com}) = 2$, $h_2(\text{cs.bristol.ac.uk}) = 4$.

Start

\[ \begin{array}{cccc} 0 & 0 & 0 & 0 \end{array} \]
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![Bloom filter example](image.png)
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---

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<th>0</th>
<th>0</th>
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<th>0</th>
</tr>
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Insert(www.bbc.co.uk)

|   | 1 | 1 | 0 | 0 |

Insert(facebook.com)

|   | 1 | 1 | 1 | 0 |
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<tr>
<td>Start</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>Insert(\text{www.bbc.co.uk})</td>
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<td>0</td>
<td>0</td>
</tr>
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<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Member(\text{cs.bristol.ac.uk})</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>returns No</td>
<td></td>
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Does the Bloom filter work?

Imagine $|S| = n$ and we query the filter with a key $k \notin S$. If a $p$ fraction of the bits of $B$ are set to 1, the probability that all of the bits checked are set to 1 is precisely $p^r$. At most $nr$ bits of $B$ can be set to 1 (each key inserted sets at most $r$ bits to 1). So the fraction of bits set to 1 is at most $nr/m$. So the probability that we incorrectly output 1 is at most $(nr/m)^r$. 
Does the Bloom filter work?

- Imagine $|S| = n$ and we query the filter with a key $k \notin S$.

- This is equivalent to checking $r$ random indices $h_1(k), \ldots, h_r(k)$ and returning Yes if all of the bits are set to 1. We now upper-bound the probability of this happening.
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- With this value of \( r \), we get that the failure probability is at most \( e^{-m/(ne)} \approx 0.69^{m/n} \).

- So, to achieve failure probability \( p \), we can choose any \( m \) such that \( e^{-m/(ne)} \leq p \), which is equivalent to

\[
m \geq -en \ln p.
\]
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- By taking the derivative, we find that the minimum of $(nr/m)^r$ is achieved when $r = m/(ne)$, where $e = 2.7818\ldots$.

- With this value of $r$, we get that the failure probability is at most $e^{-m/(ne)} \approx 0.69m/n$.

- So, to achieve failure probability $p$, we can choose any $m$ such that $e^{-m/(ne)} \leq p$, which is equivalent to

$$m \geq -en \ln p.$$ 

- For small $p$, this is much better than using one hash function. For example, to achieve $p = 0.01$ (i.e. a 1% failure probability), we can take $m \approx 12.52n$. 

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So the number of bits $m$ used by the Bloom filter is only a (small) multiple of $n$, and does not depend on $U$. 
Can we do as well deterministically?

**Claim**

Any data structure that stores a subset $S$ of $n$ elements of a universe of size $U$, in such a way that membership in $S$ can be tested with certainty, must use $\Omega(n \log U)$ bits of storage.
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- By testing membership in $S$ of each element of the universe in turn, we can determine $S$ completely, so the structure must contain enough information to identify $S$.
- **Claim:** there are at least $\lfloor U/n \rfloor^n$ subsets of $U$ of size $n$.
- **Proof:** divide $U$ into $n$ blocks of (nearly) equal size, and consider only subsets with one item in each block. There are $\lfloor U/n \rfloor^n$ such subsets.

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Lower bounds on storage space

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- Thus, unless $2^b \geq \lceil U/n \rceil^n$, there must exist two subsets that correspond to the same bit-string.
- If the structure gives the right answer for all subsets, we must have

$$b \geq \log_2(\lceil U/n \rceil^n) = n \log_2 \lceil U/n \rceil = \Omega(n \log U).$$
Practical considerations

- We made the unrealistic assumption that each hash function $h_i$ maps a key $k$ to a uniformly random integer between 1 and $m$. 

  - In practice, we would pick each hash function $h_i$ randomly from a fixed set of hash functions. One way of doing this for integer keys (see CLRS §11.3.3) is to do the following for each $i$:
    1. Pick a prime number $p > U$.
    2. Pick random integers $a \in \{1, \ldots, p-1\}$, $b \in \{0, \ldots, p-1\}$.
    3. Let $h_i$ be defined by $h_i(k) = 1 + ((ak + b) \mod p) \mod m$.

  - Some number theory can be used to prove that this set of hash functions is "pseudorandom" in some sense; however, technically they are not "random enough" for our analysis above to go through.

  - Nevertheless, in practice hash functions like this are very effective.
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Bloom filters have a number of applications: web caches, databases (e.g., Google BigTable, Apache Cassandra), spell checkers, Bitcoin (!), the Linux kernel, ... They are very efficient in theory and even more efficient in practice. There are modifications to Bloom filters to allow deletions (“counting Bloom filter”), storage of key values (“Bloomier filter”), dynamic scaling, ...
Summary

- **Bloom filters** provide a way of checking membership in a set which is very efficient in both space and time.

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Further reading

- **Probability and Computing**
  Michael Mitzenmacher and Eli Upfal
  Cambridge University Press
  - Section 5.5.3 – Bloom Filters

- **Network Applications of Bloom Filters: A Survey**
  Andrei Broder and Michael Mitzenmacher

- This year’s lecture slides for **COMS31900: Advanced Algorithms**, for additional / more advanced material.
  - Lecture 5 – Bloom filters
The Bloom filter was invented by Burtnon Howard Bloom in 1970, in a paper which now has over 4000 citations.

His analysis of the structure turned out to have a bug which was only fixed in a paper published in 2008!

Bloom is sadly lacking a Wikipedia page and online photo.