1. Assume we have a universe $U$ which consists of the integers between 1 and 8, and we want to store subsets of $U$ in a disjoint-set structure implemented using an array of linked lists with the weighted-union heuristic. Imagine $\text{MakeSet}(i)$ has been called for each integer $i$ between 1 and 8 to create 8 disjoint sets containing 1 element each. Describe a sequence of 7 Union operations which makes the largest possible total number of updates to the data of the elements in the data structure.

2. For each following weighted graph $G$, and subset $X$ of a minimum spanning tree (shown by thick lines), determine whether each subset $S$ (shown by coloured-in vertices) satisfies the cut property.

3. Run Kruskal’s and Prim’s algorithms on the following graph.

4. Prove the claim made in lecture that Prim’s algorithm outputs a tree.

5. Does the minimum spanning tree problem make sense if we allow the input graph to have negative-weight edges? Why or why not?

6. Prove the following claim made in lecture during the discussion of the cut property. Let $T$ be a spanning tree of an undirected graph $G$, and $e$ be an edge in $G$. Let $p$ be a path in $T$ between the two endpoints of $e$. Show that, for any edge $e'$ on the path $p$, if we replace $e'$ with $e$ in $T$, the resulting set $T'$ is still a spanning tree.

7. Prove that a spanning tree on a graph with $n$ vertices contains exactly $n - 1$ edges.

8. What happens if we run Kruskal’s and Prim’s algorithms on a graph which is not connected?