Skip lists and other search structures

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Introduction

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- For example, the database might be a list of students: the key might be a student ID, and the data might be everything else associated with that student (name, address, etc.).
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- For example, the database might be a list of students: the key might be a student ID, and the data might be everything else associated with that student (name, address, etc.).

- “Database” is in quotes because it is a database in the abstract sense, rather than (necessarily) a real-world database like MySQL...
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- We assume that the largest number of records ever stored in the database is \(n\).
- In general, we imagine that \(n\) is much smaller than \(U\).
Array

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</tr>
</thead>
<tbody>
<tr>
<td>Array</td>
<td>$\Theta(U)$</td>
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Insert, Delete and Find are all very efficient ($\Theta(1)$) but Successor could take time $\Theta(U)$, as we need to search through all subsequent elements in the array in turn.
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Insert(7, Alice)
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\[
\text{Insert}(3, \text{Bob}) \quad \begin{array}{ccccccc}
\ & \ & \ & \ & \ & \ & \text{Alice} \\
\end{array}
\]
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<th></th>
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returns 7
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Exercise: Does it help to store the records sorted by key?
Unsorted linked list

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---|---|---|---|---|---
Unsorted linked list | O\(n\) | \(\Theta(1)\) | O\(n\) | O\(n\) | O\(n\)

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\[
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Successor(3)

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returns 7

```
list head -> 3 Bob -> 7 Alice
```
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We store the table in an array of size $m$, where $m$ is much smaller than $U$. 

Given a key $k$, we compute a function $h(k)$ giving the position of $k$ in the array.

If $m < U$, there must exist records that hash to the same position. To deal with this situation, we have a linked list at each element of the array to store these elements.
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In the worst case, we might have $n$ records coming in which all hash to the same position. Then the complexity is no better than an unsorted linked list!

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<tr>
<td>Hash table</td>
<td>$O(n)$</td>
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A **binary tree** offers another way to store data, and to list it easily.
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Now the complexities of the various operations all depend on the height $h$ of the tree.

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<tbody>
<tr>
<td>Binary tree</td>
<td>$O(n)$</td>
<td>$O(h)$</td>
<td>$O(h)$</td>
<td>$O(h)$</td>
<td>$O(h)$</td>
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Binary tree

If the height is large, these operations are all inefficient. This can happen if keys are inserted into the tree in an unfortunate (e.g. ascending) order.

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<th>SUCCESSOR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Binary tree (worst case)</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
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In the worst case, this is even worse than an unsorted linked list.
AVL tree

As you heard in COMS11600, an AVL tree is a binary tree which is maintained such that the height of a tree containing \( n \) keys is \( O(\log n) \).

![AVL tree diagram]

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<td>AVL tree</td>
<td>( O(n) )</td>
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  - Delete: $O(\log n)$
  - Find: $O(\log n)$
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▶ We have now achieved much better complexities, but at the expense of having a more complicated data structure.
▶ Can we do better?
We will discuss a way in which we can get complexities which match the above AVL tree bounds, with a much simpler data structure, called the skip list.
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To be clear: when we perform an Insert, Delete, Find or Successor operation on a skip list, it always succeeds; but sometimes (if we are unlucky with our coin tosses) it might take a long time.
A skip list is a **linked list with shortcuts**.

Imagine we have a list containing $n$ keys in sorted order, e.g.:

```
1 → 2 → 5 → 9 → 16 → 18 → 25
```

(data omitted from the diagram for simplicity)
Skip lists

To accelerate search in this list, we attach another linked list which contains duplicates of $m < n$ of the keys in the list, e.g.:

```
1 ———> 5 ———> 18
  |        |        |
  |        v        |        |
1 ———> 2 ———> 5 ———> 9 ———> 16 ———> 18 ———> 25
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To find an element, we search in the new “shortcut” list to find the largest key smaller than the key we're looking for. Then we switch to the main list and continue to search for the key as normal.
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- The number of keys read is the sum of the number read in the main list and the number read in the shortcut list.

We then obtain a worst-case complexity of $O(\frac{m}{\sqrt{n}} + n)$, which is minimised by taking $m = \sqrt{n}$, giving a complexity of $O(\sqrt{n})$. 
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- The number of keys read is the sum of the number read in the main list and the number read in the shortcut list.
- A good way to minimise this in the worst case is to make the spacing of the $m$ elements in the shortcut list equal.

We then obtain a worst-case complexity of $O(m + n/m)$, which is minimised by taking $m = \sqrt{n}$, giving a complexity of $O(\sqrt{n})$. 
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- Each element is present in one or more lists, and all elements are present in the bottom list.
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To search, we start with the top list and follow the same procedure as with two lists.
# Searching in a skip list

**Find($k$)**

1. $i \leftarrow 1$
2. while $i \leq$ number of lists
3. scan along the $i$’th list until either $k$ is found, or the next element has key greater than $k$
4. $i \leftarrow i + 1$
Searching in a skip list

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- If we have an $L$-level list, and put $n^{i/L}$ equally spaced elements in each level $i$ between 1 and $L$, the worst-case number of elements read is

$$\sum_{i=1}^{L} \frac{n^{i/L}}{n^{(i-1)/L}} = \sum_{i=1}^{L} n^{1/L} = Ln^{1/L}.$$
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  $$
  \sum_{i=1}^{L} \frac{n^{i/L}}{n^{(i-1)/L}} = \sum_{i=1}^{L} n^{1/L} = L n^{1/L}.
  $$

- If we take $L = \log_2 n$, the total number of elements read is at most
  $2 \log_2 n$. 
Maintaining this data structure

- How should we maintain this structure under insertions and deletions?

Deletions are easy: when an element is deleted, we remove it from all levels of the list (to make this more efficient, we would use doubly linked lists).

Insertion is more tricky. When a new element comes in, we should add it to some number of levels of the list.

The problem is that, as we don’t know which elements will arrive in the future, we don’t know how many levels of the list to add it to, in order to keep the levels of the list equally spaced.

We will avoid this problem using randomisation.
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Maintaining this data structure

### Insert($k$)

1. search to find where $k$ should be inserted in the bottom level
2. insert $k$ in the bottom level
3. $r \leftarrow$ the result of tossing a fair coin
4. while $r = \text{HEADS}$
5. insert $k$ in the next level up
6. $r \leftarrow$ the result of tossing a fair coin

So with probability $1/2$, $k$ is only inserted in the main list; with probability $1/4$, it is inserted in the bottom two lists; with probability $1/8$, it is inserted in three lists; ...
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Imagine we start with the following skip list:

Now \textbf{Insert}(27) is called.
Example

First we search for where 27 should be inserted in the bottom list:
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We insert 27 in the bottom list.
Example

We toss a coin; assume the answer is HEADS. This means we insert 27 in the next level up.
Example

We toss a coin again; assume the answer is HEADS. This means we insert 27 in the next level up too.
We toss a coin again; assume the answer is TAILS. The algorithm terminates.
Probabilistic analysis

- We would like to show that, on average, the skip list has good performance.

Technical note: We assume that the elements to be inserted and deleted are chosen with no reference to the coin flips made by the algorithm.

The main tool from probability theory we will need:

Union bound

Let $E_1, \ldots, E_m$ be events, and let the probability of event $E_i$ occurring be $p_i$. Then the probability that one or more of the events $E_1$ through $E_m$ occur is at most $\sum_i p_i$. 
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The number of levels

We first show that a skip list does not have many levels, with high probability.

Claim

The probability that a skip list containing $n$ elements has $2 \log_2 n$ levels or more is at most $1/n$.
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- By the union bound, the probability of having at least $2 \log_2 n$ levels is at most $n$ times the probability that an individual element is inserted in at least $2 \log_2 n$ levels.
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- So the probability that the list has at least $2 \log_2 n$ levels is at most $n \times 1/n^2 = 1/n$. □
The search time

Claim

Let $L$ be a skip list containing $n$ elements. Then the expected time to find an element in $L$ is $O(\log n)$. 

Proof (sketch)

- We analyse the behaviour of the algorithm when searching for an item.
- Key observation: apart from the first item, we only examine an element at a given level if it was not present on the level above.
- Therefore, the expected number of elements examined on a given level is the same as the expected number of flips of a coin required until we get heads.
- This is $\frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \cdots = \sum_{i \geq 1} i \cdot 2^{-i} = 2$.
- So if there are at most $2 \log_2 n$ levels in the list, we examine an expected number of at most $4 \log_2 n$ elements.
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- On the other hand, we always examine at most $n$ elements.
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Proof (sketch)

- On the other hand, we always examine at most $n$ elements.
- So the expected number of elements examined is at most

\[
\Pr[\geq 2 \log_2 n \text{ levels}] \times n + \Pr[\leq 2 \log_2 n \text{ levels}] \times 2 \times (2 \log_2 n) \\
\leq 1/n \times n + 1 \times 2 \times (2 \log_2 n) \\
= O(\log n).
\]
A skip list is a **linked list with shortcuts**. The skip list shows how **randomness** can be useful in the design of data structures.
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A number of advanced database applications are built around skip lists, e.g. levelDB (Google).
Other search structures?

- Another interesting data structure, which achieves similar performance to skip lists, is the treap.
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Just as skip lists can be seen as randomised linked lists, treaps can be seen as randomised binary trees.

On modern computer systems, there are many other factors to be considered when choosing a search structure (e.g. performance with respect to caching and external memory, concurrent access, . . . ).
## Summary of complexities

<table>
<thead>
<tr>
<th>Structure</th>
<th>Space</th>
<th>Insert</th>
<th>Delete</th>
<th>Find</th>
<th>Successor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Array</td>
<td>$\Theta(U)$</td>
<td>$\Theta(1)$</td>
<td>$\Theta(1)$</td>
<td>$\Theta(1)$</td>
<td>$O(U)$</td>
</tr>
<tr>
<td>Unsorted linked list</td>
<td>$O(n)$</td>
<td>$\Theta(1)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Hash table</td>
<td>$O(n)$</td>
<td>$\Theta(1)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
</tr>
<tr>
<td>Binary tree</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
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<tr>
<td>AVL tree</td>
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<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
</tr>
<tr>
<td>Skip list</td>
<td>$O(n)$</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
<td>$O(\log n)$</td>
</tr>
<tr>
<td>Holy grail</td>
<td>$O(n)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
<td>$O(1)$</td>
</tr>
</tbody>
</table>

- All complexities listed are worst-case.
- The skip list is randomised; all others are deterministic.
- How close can we get to the holy grail? An object of current research!
  See COMS31900: Advanced Algorithms for more...
Further Reading

- **Introduction to Algorithms**
  - Section 10.2 – Linked lists
  - Section 11.1 – Directly addressed arrays
  - Section 11.2 – Hash tables
  - Exercise 13-3 – AVL trees
  - Exercise 13-4 – Treaps

- **Algorithms lecture notes, University of Illinois**
  Jeff Erickson
  http://www.cs.uiuc.edu/~jeffe/teaching/algorithms/
  - Lecture 10 – Treaps and skip lists
William J. Pugh

- Bill Pugh developed skip lists in the 1980s.
- Also contributions to programming language design and implementation, including the Java memory model.
- Currently a professor emeritus at the University of Maryland.