QUANTUM COMPUTATION

Exercise sheet 2

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1. **A simple case of Grover's algorithm.** Consider the unstructured search problem in the 4-element set $\{0,1\}^2$, where there is a unique marked element $x_0 = 10$.

   (a) Write down the matrix for the oracle operation U_f for this value of x_0, with respect to the computational basis.

   (b) Write down the matrix for the operator D occurring in Grover's algorithm for $N = 4$, with respect to the computational basis.

   (c) Multiply out all the matrices and vectors occurring for one step of Grover's algorithm, to verify the claim in the lecture notes that the algorithm finds the marked element with certainty.

   (d) What is the final state if another step is made? What is the probability that the marked element is found if this state is measured?

2. **Grover's algorithm for larger input sizes.** Here we consider Grover's algorithm with $N = 2^n$ for arbitrary integer n, and a unique marked element x_0.

   (a) Calculate an expression for $\langle x|D|y\rangle$ for arbitrary $x, y \in \{0,1\}^n$.

   (b) What is the probability that the marked element is found if the qubits are measured after only one step of the algorithm? Is this larger or smaller than we can achieve using a classical algorithm that makes one query?

3. **Grover's algorithm with errors.** Imagine we are attempting to run Grover's algorithm, but with an oracle U_f which sometimes fails to work. That is, sometimes it does nothing on every state (performs the identity operation), rather than the intended operation of mapping $|x_0\rangle$ to $-|x_0\rangle$, and leaving all other states $|x\rangle$ unchanged.

   (a) What is the effect on the algorithm if the oracle fails the first time it is used, but works on all subsequent occasions?

   (b) Assume that the overall algorithm makes T uses of U_f. What is the effect on the algorithm if the oracle fails the $\lceil T/2 \rceil$'th time it is used, but works on all other occasions?

4. **Running Grover's algorithm on a real quantum computer (optional).**

The IBM Quantum Experience (`www.research.ibm.com/quantum/`) is a web interface to a real quantum computing experiment at IBM. In this question you will use it to implement and run Grover's algorithm for $N = 4$ and a unique marked element x_0.

(a) Sign up for an account on the IBM Quantum Experience website, log in and familiarise yourself with the interface. Note that the site uses "score" to mean "circuit".

(b) Find quantum circuits containing only H, X and CNOT gates for the operations U_0 and U_f occurring in Grover's algorithm, for the four different possible values for x_0. Hint: exercise sheet 1, Q1c could be useful.

(c) Use your answer to part (b) to implement the quantum circuits corresponding to one iteration of Grover's algorithm, for these different possible choices for x_0, on the IBM Quantum Experience. (Note that a $-I$ gate is not available, so replace D with $-D$ in the definition of the algorithm.) Run the circuits on the simulator. Check that the answer is what you expect.

(d) Run the same circuits on the real quantum computer. Note that this has restrictions on the allowed topology of gates (e.g. CNOT gates must have the third qubit as target). Is the answer what you expect?

(e) Work through the user guide for the IBM Quantum Experience and experiment with other circuits (for example, Grover's algorithm with $N = 8$).