1. **Shor's 9 qubit code.** Imagine we encode the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ using Shor's 9 qubit code, and then an $X$ error occurs on the 8th qubit of the encoded state $|E(\psi)\rangle$.

(a) Write down the state following the error.

**Answer:**

$$\frac{1}{2\sqrt{2}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|010\rangle + |101\rangle)$$

$$+ \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|010\rangle - |101\rangle).$$

(b) We now decode the encoded state, starting by applying the bit-flip code decoding algorithm. What are the syndromes returned by the measurements in the algorithm?

**Answer:** Using the table in the lecture notes, the syndromes are 00, 00, 10.

(c) Now imagine that $|E(\psi)\rangle$ is affected by two $X$ errors, on the 7th and 8th qubits. What are the syndromes returned this time? What state does the decoding algorithm output?

**Answer:** Now the syndromes are 00, 00, 01. The decoding algorithm thus thinks there has been an $X$ error on the 9th qubit. So it “corrects” this by applying an $X$ operation on this qubit, to give the state

$$\frac{1}{2\sqrt{2}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$- \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle).$$

Note that $\beta$ now has a minus sign in front of it. After the bit-flip decoding, we are left with $\alpha|++-\rangle - \beta|--\rangle$, which is then decoded to $\alpha|0\rangle - \beta|1\rangle$.

(d) Which patterns of $X$ errors are corrected by Shor’s 9 qubit code?

**Answer:** If there is at most one $X$ error in each block of 3 qubits, these will be corrected properly. We have just seen that, if two errors occur in one block,
the sign of $\beta$ will be flipped, but the state is not otherwise affected; a similar argument holds for 3 errors in one block. So the output state will be correct if the number of blocks in which at least two errors occur is even (as then $\beta$ will eventually be left unchanged).

2. Stabilizers.

(a) Show that $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is stabilized by $\{-X \otimes X, -Z \otimes Z\}$.

\textbf{Answer:} Direct calculation: $(X \otimes X)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) = -\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and similarly for $Z \otimes Z$.

(b) Show that $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is a stabilizer state and write down its stabilizer.

\textbf{Answer:} This can be shown either by experimenting with Pauli matrices on 2 qubits, or using the fact that $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = (I \otimes X)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, so the stabilizer of $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ must be the same as that of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ conjugated by $I \otimes X$. The final answer is $\{X \otimes X, -Z \otimes Z\}$.

(c) List all the stabilizer states of one qubit.

\textbf{Answer:} These can be determined by considering the eigenvectors of the Pauli matrices. Up to overall phases, the states are:

<table>
<thead>
<tr>
<th>State</th>
<th>Stabilizer</th>
</tr>
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<tbody>
<tr>
<td>$</td>
<td>0\rangle$</td>
</tr>
<tr>
<td>$</td>
<td>1\rangle$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{2}}(</td>
<td>0\rangle +</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{2}}(</td>
<td>0\rangle -</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{2}}(</td>
<td>0\rangle + i</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{2}}(</td>
<td>0\rangle - i</td>
</tr>
</tbody>
</table>

(d) Prove the claim in the lecture notes that every pair of Pauli matrices on $n$ qubits, i.e. matrices of the form $M = M_1 \otimes M_2 \otimes \cdots \otimes M_n$, where for each $i$, $M_i \in \{I, X, Y, Z\}$, either commutes or anticommutes.

\textbf{Answer:} It can be shown by direct calculation that every pair of Pauli matrices on one qubit either commutes or anticommutes (e.g. $XY = -YX$). Let $M$ and $M'$ be Pauli matrices on $n$ qubits. We have

$$MM' = (M_1M'_1) \otimes (M_2M'_2) \otimes \cdots \otimes (M_nM'_n)$$
and, for all $i$, $M_i M'_i = \pm M'_i M_i$. Multiplying out the signs, $MM' = \pm M'M$, so $M$ and $M'$ either commute or anticommute.