School of Mathematics **Spring 2017**

QUANTUM COMPUTATION

Exercise sheet 6

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1. **Shor's 9 qubit code.** Imagine we encode the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ using Shor's 9 qubit code, and then an X error occurs on the 8th qubit of the encoded state $|E(\psi)\rangle$.

   (a) Write down the state following the error.

   Answer:

$$\frac{1}{2\sqrt{2}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|010\rangle + |101\rangle) \\ + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|010\rangle - |101\rangle)).$$

   (b) We now decode the encoded state, starting by applying the bit-flip code decoding algorithm. What are the syndromes returned by the measurements in the algorithm?

   Answer: Using the table in the lecture notes, the syndromes are 00, 00, 10.

   (c) Now imagine that $|E(\psi)\rangle$ is affected by two X errors, on the 7th and 8th qubits. What are the syndromes returned this time? What state does the decoding algorithm output?

   Answer: Now the syndromes are 00, 00, 01. The decoding algorithm thus thinks there has been an X error on the 9th qubit. So it "corrects" this by applying an X operation on this qubit, to give the state

$$\frac{1}{2\sqrt{2}}(\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \\ - \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)).$$

   Note that β now has a minus sign in front of it. After the bit-flip decoding, we are left with $\alpha|+++\rangle - \beta|---\rangle$, which is then decoded to $\alpha|0\rangle - \beta|1\rangle$.

   (d) Which patterns of X errors are corrected by Shor's 9 qubit code?

   Answer: If there is at most one X error in each block of 3 qubits, these will be corrected properly. We have just seen that, if two errors occur in one block, the sign of β will be flipped, but the state is not otherwise affected; a similar argument holds for 3 errors in one block. So the output state will be correct if the number of blocks in which at least two errors occur is even (as then β will eventually be left unchanged).

2. **Stabilizers.**

(a) Show that $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ is stabilized by $\{-X \otimes X, -Z \otimes Z\}$.

Answer: Direct calculation: $(X \otimes X)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) = -\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and similarly for $Z \otimes Z$.

(b) Show that $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is a stabilizer state and write down its stabilizer.

Answer: This can be shown either by experimenting with Pauli matrices on 2 qubits, or using the fact that $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = (I \otimes X)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, so the stabilizer of $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ must be the same as that of $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ conjugated by $I \otimes X$. The final answer is $\{X \otimes X, -Z \otimes Z\}$.

(c) List all the stabilizer states of one qubit.

Answer: These can be determined by considering the eigenvectors of the Pauli matrices. Up to overall phases, the states are:

State	**Stabilizer**
$\|0\rangle$	$\{I, Z\}$
$\|1\rangle$	$\{I, -Z\}$
$\frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$\{I, X\}$
$\frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$\{I, -X\}$
$\frac{1}{\sqrt{2}}(\|0\rangle + i\|1\rangle)$	$\{I, Y\}$
$\frac{1}{\sqrt{2}}(\|0\rangle - i\|1\rangle)$	$\{I, -Y\}$

(d) Prove the claim in the lecture notes that every pair of Pauli matrices on n qubits, i.e. matrices of the form

$$M = M_1 \otimes M_2 \otimes \cdots \otimes M_n,$$

where for each i, $M_i \in \{I, X, Y, Z\}$, either commutes or anticommutes.

Answer: It can be shown by direct calculation that every pair of Pauli matrices on one qubit either commutes or anticommutes (e.g. $XY = -YX$). Let M and M' be Pauli matrices on n qubits. We have

$$MM' = (M_1 M_1') \otimes (M_2 M_2') \otimes \cdots \otimes (M_n M_n')$$

and, for all i, $M_i M_i' = \pm M_i' M_i$. Multiplying out the signs, $MM' = \pm M'M$, so M and M' either commute or anticommute.