This sample paper contains ONE question (so should be completed in ~45 minutes). The final exam will contain TWO questions. All answers will be used for assessment.

Calculators are not permitted in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.
1. Parts (a) and (b) of this question are not related.

(a) This part is about a quantum algorithm described by the following circuit on 2 qubits:

\[
\begin{array}{c}
|0\rangle & \xrightarrow{H} & |0\rangle \\
|0\rangle & \xrightarrow{U} & |0\rangle \\
|0\rangle & \xrightarrow{H} & |0\rangle \\
\end{array}
\]

where \( H \) is the Hadamard gate defined by \( H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \) with respect to the computational basis, and \( U \) is an arbitrary unitary operator.

i. (5 marks.) Assume that we are promised that either \( U = X \) or \( U = Y \), where \( X \) and \( Y \) are the unitary operators defined by the matrices \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and \( Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) with respect to the computational basis. Calculate the state output by the circuit in each case. Hence show that the circuit can be used to determine the identity of \( U \) with certainty.

ii. (2 marks.) Let \( U_\theta \) be the operator defined by \( U_\theta = (\sin \theta)X + (\cos \theta)Y \). Write down \( U_\theta \) as a matrix with respect to the computational basis and hence, or otherwise, show that \( U_\theta \) is unitary.

iii. (2 marks.) Now imagine that \( U = U_\theta \) for some \( \theta \). Write down the state output by the circuit in terms of \( \theta \).

iv. (4 marks.) Now assume that we are promised that either \( \theta = \pi/4 \) or \( \theta = 3\pi/4 \), corresponding to either \( U = \frac{1}{\sqrt{2}}(X + Y) \) or \( U = \frac{1}{\sqrt{2}}(X - Y) \). Describe how the algorithm above can be modified such that \( U \) can be identified with certainty, using gates picked from the set \( \{H, T, X, Y, Z\} \), where \( T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \) with respect to the computational basis. Draw a quantum circuit for your new algorithm.

(b) This part is about Grover’s algorithm for search for a unique marked element \( x_0 \) in a set of size \( N = 2^n \).

One iteration of Grover’s algorithm consists of applying the unitary operation \( DU_f \) on \( n \) qubits, initially in the state \( |+\rangle = \frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle \), where: \( D = -H^\otimes n U_0 H^\otimes n \); \( U_0 \) is defined by \( U_0|0^n\rangle = -|0^n\rangle \), \( U_0|x\rangle = |x\rangle \) for \( x \neq 0^n \); and \( U_f|x_0\rangle = -|x_0\rangle \), \( U_f|x\rangle = |x\rangle \) for \( x \neq x_0 \).

i. (2 marks.) Show that, for an arbitrary quantum state \( |\psi\rangle \), the operator \( R_{|\psi\rangle} = 2|\psi\rangle\langle\psi| - I \) is unitary.

ii. (2 marks.) Write \( D \) and \( U_f \) in terms of operators of the form \( R_{|\psi\rangle} \) for different states \( |\psi\rangle \).

iii. (4 marks.) Briefly describe how one iteration of Grover’s algorithm can be interpreted as a rotation in a 2d plane.

iv. (4 marks.) Determine an expression for the angle of this rotation in terms of \( N \). Hence argue that Grover’s algorithm can find \( x_0 \) with high probability using each of \( D \) and \( U_f \) \( O(\sqrt{N}) \) times.

End of examination.