# QUANTUM COMPUTATION

## Exercise sheet 5

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1. **More efficient quantum simulation.**

   (a) Let $A$ and $B$ be Hermitian operators with $\|A\| \leq \delta$, $\|B\| \leq \delta$ for some $\delta \leq 1$. Show that
   $$e^{-iA/2}e^{-iB}e^{-iA/2} = e^{-i(A+B)} + O(\delta^3)$$
   (this is the so-called *Strang splitting*). Use this to give a more efficient quantum algorithm for simulating $k$-local Hamiltonians than the algorithm discussed in the lecture, and calculate its complexity.

   **Answer:**
   $$\begin{aligned} &e^{-iA/2}e^{-iB}e^{-iA/2} \\ &= \left(I - iA/2 - A^2/8 + O(\delta^3)\right)\left(I - iB - B^2/2 + O(\delta^3)\right)\left(I - iA/2 - A^2/8 + O(\delta^3)\right) \\ &= I - iA - iB - A^2/2 - AB/2 - BA/2 - B^2/2 + O(\delta^3) \\ &= I - iA - iB - (A+B)^2/2 + O(\delta^3) \\ &= e^{-i(A+B)} + O(\delta^3). \end{aligned}$$

   Plugging this in to the argument of the lecture notes, for operators $H_1, H_2, \dots, H_m$ such that $\|H_i\| \leq \delta$ we obtain
   $$e^{-iH_1/2}e^{-iH_2/2}\dots e^{-iH_m}e^{-iH_{m-1}/2}\dots e^{-iH_1/2} = e^{-i(H_1+\dots+H_m)} + O(m^4\delta^3).$$

   So, for some universal constant $C$, if $p \geq Cm^2(t\delta)^{3/2}/\epsilon^{1/2}$,
   $$\left\|\left(e^{-iH_1t/(2p)}e^{-iH_2t/(2p)}\dots e^{-iH_mt/p}e^{-iH_{m-1}t/(2p)}\dots e^{-iH_1t/(2p)}\right)^p - e^{-i(H_1+\dots+H_m)t}\right\| \leq \epsilon.$$

   Thus a $k$-local Hamiltonian which is a sum of $m$ terms $H_1, \dots, H_m$, where $\|H_i\| \leq 1$, can be simulated for time $t$ in $O(m^3t^{3/2}/\epsilon^{1/2})$ steps.

   (b) Let $H$ be a Hamiltonian on $n$ qubits which can be written as $H = UDU^\dagger$, where $U$ is a unitary matrix that can be implemented by a quantum circuit running in time poly$(n)$, and $D = \sum_x d(x)|x\rangle\langle x|$ is a diagonal matrix such that the map $|x\rangle \mapsto e^{-id(x)t}|x\rangle$ can be implemented in time poly$(n)$ for all $x$. Show that $e^{-iHt}$ can be implemented in time poly$(n)$.

**Answer:** By linearity, the unitary operator which performs the map $|x\rangle \mapsto e^{-id(x)t}|x\rangle$ is equal to the matrix $e^{-iDt}$. And by the identity

$$U^\dagger e^{-iDt} U = e^{-iU^\dagger DUt} = e^{-iHt},$$

performing $U$, then $e^{-iDt}$, then $U^\dagger$, suffices to implement $e^{-iHt}$. Each of these steps can be carried out in time poly($n$).

2. **The amplitude damping channel.** The amplitude damping channel $\mathcal{E}_{\mathrm{AD}}$ has Kraus operators (with respect to the standard basis)

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

for some $\gamma$.

(a) What is the result of applying the amplitude damping channel to the pure state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$?
**Answer:** Using the Kraus operators, we can calculate directly that the answer is the mixed state

$$\frac{1}{2}\begin{pmatrix} 1+\gamma & \sqrt{1-\gamma} \\ \sqrt{1-\gamma} & 1-\gamma \end{pmatrix}.$$

(b) Show that, when applied to the Pauli matrices $X$, $Y$, $Z$, $\mathcal{E}_{\mathrm{AD}}$ rescales each one by a factor depending on $\gamma$, and determine what these factors are.
**Answer:** Using the Kraus operators again, we get

$$\mathcal{E}_{\mathrm{AD}}(X) = \sqrt{1-\gamma}X, \quad \mathcal{E}_{\mathrm{AD}}(Y) = \sqrt{1-\gamma}Y, \quad \mathcal{E}_{\mathrm{AD}}(Z) = (1-\gamma)Z$$

by direct calculation.

(c) Hence determine the representation of the amplitude-damping channel as an affine map $v \mapsto Av + b$ on the Bloch sphere.
**Answer:** We can calculate

$$\mathcal{E}_{\mathrm{AD}}\left(\frac{I}{2}\right) = E_0 \frac{I}{2} E_0^\dagger + E_1 \frac{I}{2} E_1^\dagger = \frac{1}{2}\begin{pmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{pmatrix} = \frac{1}{2}(I + \gamma Z),$$

which tells us that $b = (0, 0, \gamma)$. We can then use the previous question to determine $A$ by writing down the columns of $A$, with respect to the standard basis, in terms of the coefficients obtained there.

$$A = \begin{pmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix}.$$

(d) What does this channel "look like" geometrically in terms of its effect on the Bloch sphere? **Answer:** The channel shrinks the $X$ and $Y$ directions, but leaves the $Z$ direction unchanged. A vector of the form $(x, y, z)$ is mapped to $(\sqrt{1-\gamma}x, \sqrt{1-\gamma}y, z)$. So the amplitude-damping channel squeezes the Bloch sphere into an ellipsoid which is no longer centred on $I/2$.

3. **General quantum channels.**

(a) Given two channels $\mathcal{E}_1$, $\mathcal{E}_2$, with Kraus operators $\{E_k^{(1)}\}$, $\{E_k^{(2)}\}$, what is the Kraus representation of the composite channel $\mathcal{E}_2 \circ \mathcal{E}_1$ which is formed by first applying $\mathcal{E}_1$, then applying $\mathcal{E}_2$?

**Answer:** The output of the composite channel applied to $\rho$ is

$$\begin{aligned}(\mathcal{E}_2 \circ \mathcal{E}_1)(\rho) &= \mathcal{E}_2\left(\sum_j E_j^{(1)}\rho(E_j^{(1)})^\dagger\right) = \sum_k E_k^{(2)}\left(\sum_j E_j^{(1)}\rho(E_j^{(1)})^\dagger\right)(E_k^{(2)})^\dagger \\ &= \sum_{j,k} E_k^{(2)}E_j^{(1)}\rho(E_j^{(1)})^\dagger(E_k^{(2)})^\dagger,\end{aligned}$$

so the Kraus operators are all products of the Kraus operators of $\mathcal{E}_1$ and $\mathcal{E}_2$, i.e. $\{E_k^{(2)}E_j^{(1)}\}$.

(b) Determine a Kraus representation for the channel Tr which maps $\rho \mapsto \operatorname{tr}\rho$ for a mixed quantum state $\rho$ in $d$ dimensions.

**Answer:** The channel has $d$ Kraus operators, $E_k = \langle k|$:

$$\operatorname{Tr}(\rho) = \sum_k \langle k|\rho|k\rangle = \operatorname{tr}\rho.$$

(c) Let $\mathcal{E}$ and $\mathcal{F}$ be quantum channels with $d$ Kraus operators each, $E_k$ and $F_k$ (respectively), such that for all $j$, $F_j = \sum_{k=1}^d U_{jk}E_k$ for some unitary matrix $U$. Show that $\mathcal{E}$ and $\mathcal{F}$ are actually the same quantum channel.

**Answer:** We have

$$\begin{aligned}\mathcal{F}(\rho) &= \sum_j F_j\rho F_j^\dagger = \sum_j\left(\sum_k U_{jk}E_k\right)\rho\left(\sum_\ell U_{j\ell}^*E_\ell^\dagger\right) \\ &= \sum_{k,\ell} E_k\rho E_\ell^\dagger \sum_k U_{jk}U_{j\ell}^* = \sum_k E_k\rho E_k^\dagger = \mathcal{E}(\rho),\end{aligned}$$

where we use unitarity of $U$.