1. More efficient quantum simulation.

(a) Let $A$ and $B$ be Hermitian operators with $\|A\| \leq \delta$, $\|B\| \leq \delta$ for some $\delta \leq 1$. Show that

$$e^{-iA/2}e^{-iB}e^{-iA/2} = e^{-i(A+B)} + O(\delta^3)$$

(this is the so-called *Strang splitting*). Use this to give a more efficient quantum algorithm for simulating $k$-local Hamiltonians than the algorithm discussed in the lecture, and calculate its complexity.

(b) Let $H$ be a Hamiltonian on $n$ qubits which can be written as $H = UDU^\dagger$, where $U$ is a unitary matrix that can be implemented by a quantum circuit running in time $\text{poly}(n)$, and $D = \sum_x d(x)|x\rangle\langle x|$ is a diagonal matrix such that the map $|x\rangle \mapsto e^{-id(x)t}|x\rangle$ can be implemented in time $\text{poly}(n)$ for all $x$. Show that $e^{-iHt}$ can be implemented in time $\text{poly}(n)$.

2. The amplitude damping channel. The amplitude damping channel $\mathcal{E}_\text{AD}$ has Kraus operators (with respect to the standard basis)

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

for some $\gamma$.

(a) What is the result of applying the amplitude damping channel to the pure state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$?

(b) Show that, when applied to the Pauli matrices $X$, $Y$, $Z$, $\mathcal{E}_\text{AD}$ rescales each one by a factor depending on $\gamma$, and determine what these factors are.

(c) Hence determine the representation of the amplitude-damping channel as an affine map $v \mapsto Av + b$ on the Bloch sphere.

(d) What does this channel “look like” geometrically in terms of its effect on the Bloch sphere?

3. General quantum channels.
(a) Given two channels $\mathcal{E}_1, \mathcal{E}_2$, with Kraus operators $\{E_k^{(1)}\}, \{E_k^{(2)}\}$, what is the Kraus representation of the composite channel $\mathcal{E}_2 \circ \mathcal{E}_1$ which is formed by first applying $\mathcal{E}_1$, then applying $\mathcal{E}_2$?

(b) Determine a Kraus representation for the channel $\text{Tr}$ which maps $\rho \mapsto \text{tr} \rho$ for a mixed quantum state $\rho$ in $d$ dimensions.

(c) Let $\mathcal{E}$ and $\mathcal{F}$ be quantum channels with $d$ Kraus operators each, $E_k$ and $F_k$ (respectively), such that for all $j$, $F_j = \sum_{k=1}^d U_{jk} E_k$ for some unitary matrix $U$. Show that $\mathcal{E}$ and $\mathcal{F}$ are actually the same quantum channel.