1. **Shor’s 9 qubit code.** Imagine we encode the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ using Shor’s 9 qubit code, and then an $X$ error occurs on the 8th qubit of the encoded state $|E(\psi)\rangle$.

(a) Write down the state following the error.

**Answer:**

$$\frac{1}{2\sqrt{2}} (\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|010\rangle + |101\rangle) + \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|010\rangle - |101\rangle)).$$

(b) We now decode the encoded state, starting by applying the bit-flip code decoding algorithm. What are the syndromes returned by the measurements in the algorithm?

**Answer:** Using the table in the lecture notes, the syndromes are 00, 00, 10.

(c) Now imagine that $|E(\psi)\rangle$ is affected by two $X$ errors, on the 7th and 8th qubits. What are the syndromes returned this time? What state does the decoding algorithm output?

**Answer:** Now the syndromes are 00, 00, 01. The decoding algorithm thus thinks there has been an $X$ error on the 9th qubit. So it “corrects” this by applying an $X$ operation on this qubit, to give the state

$$\frac{1}{2\sqrt{2}} (\alpha(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) - \beta(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)).$$

Note that $\beta$ now has a minus sign in front of it. After the bit-flip decoding, we are left with $\alpha|++\rangle - \beta|--\rangle$, which is then decoded to $\alpha|0\rangle - \beta|1\rangle$.

(d) Which patterns of $X$ errors are corrected by Shor’s 9 qubit code?

**Answer:** If there is at most one $X$ error in each block of 3 qubits, these will be corrected properly. We have just seen that, if two errors occur in one block, the sign of $\beta$ will be flipped, but the state is not otherwise affected; a similar argument holds for 3 errors in one block. So the output state will be correct if the number of blocks in which at least two errors occur is even (as then $\beta$ will eventually be left unchanged).