School of Mathematics Spring 2020

QUANTUM COMPUTATION

Exercise sheet 1

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1. **Revision.**

   (a) Imagine we have a quantum state $|\psi\rangle$ of n qubits, where $|\psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$, and we measure the first qubit of $|\psi\rangle$ in the computational basis. What is the probability that the measurement outcome is 1, in terms of the α_x coefficients?

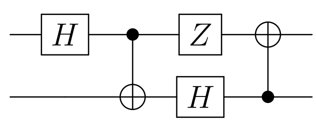
   (b) What is the state of the system after the measurement?

   (c) Let M be the matrix defined by $M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ -1 & 1 \end{pmatrix}$. Is M unitary?

   (d) Write down the matrix corresponding to the operator $H \otimes H$, in the computational basis, where H is the Hadamard operator.

2. **The quantum circuit model.**

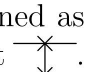
   (a) Consider the following quantum circuit C:



   i. Calculate the matrix of the unitary operation U corresponding to C, with respect to the computational basis.

   ii. Write down a quantum circuit corresponding to the inverse operation U^{-1}.

   iii. If C is applied to the initial state $|0\rangle|0\rangle$ and is followed by a measurement of each qubit in the computational basis, what is the distribution on measurement outcomes?

   (b) The SWAP gate for 2 qubits is defined as $\text{SWAP}|x\rangle|y\rangle = |y\rangle|x\rangle$ for $x, y \in \{0,1\}$ and is denoted by the circuit element . Show that SWAP can be implemented as a product of CNOT gates and write down the corresponding circuit.

   (c) Show that a CZ gate can be implemented using a CNOT gate and Hadamard gates and write down the corresponding circuit.

   (d) The classical OR gate takes as input a pair of bits $x, y \in \{0,1\}$ and outputs 1 if either x or y is equal to 1, and 0 otherwise. Use the generic construction of reversible functions discussed in the lecture notes to write down a unitary operation on 3 qubits which corresponds to a reversible implementation of the OR gate.

3. **The Bernstein-Vazirani algorithm.**

A parity function $f_s : \{0,1\}^n \to \{0,1\}$, for some $s \in \{0,1\}^n$, is a function of the form $f_s(x) = x \cdot s$, where the inner product is taken modulo 2. For example, with $n = 3$, $f_{110}(x)$ is the function $x_1 \oplus x_2$.

(a) Show that f_s is a balanced function for all $s \neq 0^n$.

(b) Imagine we apply the circuit for the Deutsch-Jozsa algorithm with the oracle U_{f_s}. Show that the measured output is precisely the string s.

(c) Consider the following problem: given oracle access to a parity function f_s, determine s using the minimal number of queries to f_s.

i. Conclude from (b) that there is a quantum algorithm that solves this problem with one query to f_s.

ii. Give an exact bound on the number of queries to f_s required for a classical algorithm to solve the problem with certainty.

4. **Simulation of various kinds. (Optional)**

(a) Show that the phase oracle U_f as defined in the lecture notes cannot be used to implement the bit oracle O_f in general, even if f only has 1 bit output.

(b) Imagine we are given a quantum circuit on n qubits which consists of poly(n) gates picked from the (universal) set $\{H, X, \mathrm{CNOT}, T\}$, followed by a final measurement of all the qubits. Assume that at each step in the computation the quantum state is unentangled (i.e. is a product state of the n qubits). Show that the circuit can be simulated efficiently classically: that is, there is an efficient classical algorithm for exactly sampling from the probability distribution on the final measurement outcomes.