QUANTUM COMPUTATION

Exercise sheet 5

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1. **Pauli matrices.**

(a) Show that any 2×2 matrix M can be written as $M = \alpha I + \beta X + \gamma Y + \delta Z$ for some coefficients $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

Answer: We have

$$\alpha I + \beta X + \gamma Y + \delta Z = \begin{pmatrix} \alpha + \delta & \beta - i\gamma \\ \beta + i\gamma & \alpha - \delta \end{pmatrix}.$$

For any matrix

$$M = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

we can set $\alpha = (w+z)/2$, $\beta = (x+y)/2$, $\gamma = i(x-y)/2$, $\delta = (w-z)/2$ to achieve the required equality.

(b) For $s \in \{I, X, Y, Z\}^n$, let σ_s denote the matrix which is a tensor product of the corresponding Pauli matrices, $\sigma_s = s_1 \otimes s_2 \otimes \cdots \otimes s_n$. Using part (a), show that any $2^n \times 2^n$ matrix M can be written as

$$M = \sum_{s \in \{I,X,Y,Z\}^n} \alpha_s \sigma_s$$

for some coefficients $\alpha_s \in \mathbb{C}$.

Answer: We can write $M = \sum_{x,y \in \{0,1\}^n} \alpha_{xy} |x\rangle\langle y|$. Each term $|x\rangle\langle y|$ is a tensor product of the form $|x_1\rangle\langle y_1| \otimes \cdots \otimes |x_n\rangle\langle y_n|$. So M can be expressed as claimed by writing each $|x_i\rangle\langle y_i|$ as a weighted sum of Pauli matrices by part (a), and expanding.

(c) Show that if M is Hermitian, we can assume that $\alpha_s \in \mathbb{R}$. [Hint: consider $\frac{1}{2}(M + M^\dagger)$.]

Answer: From the previous part, $M = \sum_{s \in \{I,X,Y,Z\}^n} \alpha_s \sigma_s$ and hence $M^\dagger = \sum_{s \in \{I,X,Y,Z\}^n} \alpha_s^* \sigma_s$. If M is Hermitian, $M = M^\dagger$. So $M = \frac{1}{2}(M + M^\dagger) = \sum_{s \in \{I,X,Y,Z\}^n} \frac{1}{2}(\alpha_s + \alpha_s^*) \sigma_s = \sum_{s \in \{I,X,Y,Z\}^n} \mathrm{Re}(\alpha_s) \sigma_s$.

2. **More efficient quantum simulation.**

(a) Let A and B be Hermitian operators with $\|A\| \leq \delta$, $\|B\| \leq \delta$ for some $\delta \leq 1$. Show that

$$e^{-iA/2}e^{-iB}e^{-iA/2} = e^{-i(A+B)} + O(\delta^3)$$

(this is the so-called *Strang splitting*). Use this to give a more efficient quantum algorithm for simulating k-local Hamiltonians than the algorithm discussed in the lecture, and calculate its complexity.

Answer:

$$\begin{aligned} & e^{-iA/2}e^{-iB}e^{-iA/2} \\ &= \left(I - iA/2 - A^2/8 + O(\delta^3)\right)\left(I - iB - B^2/2 + O(\delta^3)\right)\left(I - iA/2 - A^2/8 + O(\delta^3)\right) \\ &= I - iA - iB - A^2/2 - AB/2 - BA/2 - B^2/2 + O(\delta^3) \\ &= I - iA - iB - (A+B)^2/2 + O(\delta^3) \\ &= e^{-i(A+B)} + O(\delta^3). \end{aligned}$$

Plugging this in to the argument of the lecture notes, for operators $H_1, H_2, \dots, H_m$ such that $\|H_i\| \leq \delta$ we obtain

$$e^{-iH_1/2}e^{-iH_2/2}\dots e^{-iH_m}e^{-iH_{m-1}/2}\dots e^{-iH_1/2} = e^{-i(H_1+\dots+H_m)} + O(m^4\delta^3).$$

So, for some universal constant C, if $p \geq Cm^2(t\delta)^{3/2}/\epsilon^{1/2}$,

$$\left\|\left(e^{-iH_1t/(2p)}e^{-iH_2t/(2p)}\dots e^{-iH_mt/p}e^{-iH_{m-1}t/(2p)}\dots e^{-iH_1t/(2p)}\right)^p - e^{-i(H_1+\dots+H_m)t}\right\| \leq \epsilon.$$

Thus a k-local Hamiltonian which is a sum of m terms $H_1, \dots, H_m$, where $\|H_i\| \leq 1$, can be simulated for time t in $O(m^3t^{3/2}/\epsilon^{1/2})$ steps.

(b) Let H be a Hamiltonian on n qubits which can be written as $H = UDU^\dagger$, where U is a unitary matrix that can be implemented by a quantum circuit running in time poly(n), and $D = \sum_x d(x)|x\rangle\langle x|$ is a diagonal matrix such that the map $|x\rangle \mapsto e^{-id(x)t}|x\rangle$ can be implemented in time poly(n) for all x. Show that e^{-iHt} can be implemented in time poly(n).

Answer: By linearity, the unitary operator which performs the map $|x\rangle \mapsto e^{-id(x)t}|x\rangle$ is equal to the matrix e^{-iDt}. And by the identity

$$U^\dagger e^{-iDt}U = e^{-iU^\dagger DUt} = e^{-iHt},$$

performing U, then e^{-iDt}, then $U^\dagger$, suffices to implement e^{-iHt}. Each of these steps can be carried out in time poly(n).

3. **The amplitude damping channel.** The amplitude damping channel $\mathcal{E}_{\mathrm{AD}}$ has Kraus operators (with respect to the standard basis)

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

for some γ.

(a) What is the result of applying the amplitude damping channel to the pure state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$?

Answer: Using the Kraus operators, we can calculate directly that the answer is the mixed state

$$\frac{1}{2}\begin{pmatrix} 1+\gamma & \sqrt{1-\gamma} \\ \sqrt{1-\gamma} & 1-\gamma \end{pmatrix}.$$

(b) Show that, when applied to the Pauli matrices X, Y, Z, $\mathcal{E}_{\mathrm{AD}}$ rescales each one by a factor depending on γ, and determine what these factors are.

Answer: Using the Kraus operators again, we get

$$\mathcal{E}_{\mathrm{AD}}(X) = \sqrt{1-\gamma}X, \quad \mathcal{E}_{\mathrm{AD}}(Y) = \sqrt{1-\gamma}Y, \quad \mathcal{E}_{\mathrm{AD}}(Z) = (1-\gamma)Z$$

by direct calculation.

(c) Hence determine the representation of the amplitude-damping channel as an affine map $v \mapsto Av + b$ on the Bloch sphere.

Answer: We can calculate

$$\mathcal{E}_{\mathrm{AD}}\left(\frac{I}{2}\right) = E_0\frac{I}{2}E_0^\dagger + E_1\frac{I}{2}E_1^\dagger = \frac{1}{2}\begin{pmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{pmatrix} = \frac{1}{2}(I+\gamma Z),$$

which tells us that $b = (0, 0, \gamma)$. We can then use the previous question to determine A by writing down the columns of A, with respect to the standard basis, in terms of the coefficients obtained there.

$$A = \begin{pmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix}.$$

(d) What does this channel "look like" geometrically in terms of its effect on the Bloch sphere?

Answer: The channel shrinks the X and Y directions, and squeezes and shifts the Z direction. A vector of the form (x, y, z) is mapped to $(\sqrt{1-\gamma}x, \sqrt{1-\gamma}y, \gamma + (1-\gamma)z)$. So the amplitude-damping channel squeezes the Bloch sphere into an ellipsoid which is no longer centred on $I/2$.

4. **General quantum channels.**

(a) Given two channels $\mathcal{E}_1$, $\mathcal{E}_2$, with Kraus operators $\{E_k^{(1)}\}$, $\{E_k^{(2)}\}$, what is the Kraus representation of the composite channel $\mathcal{E}_2 \circ \mathcal{E}_1$ which is formed by first applying $\mathcal{E}_1$, then applying $\mathcal{E}_2$?

Answer: The output of the composite channel applied to ρ is

$$\begin{aligned}(\mathcal{E}_2 \circ \mathcal{E}_1)(\rho) &= \mathcal{E}_2\left(\sum_j E_j^{(1)}\rho(E_j^{(1)})^\dagger\right) = \sum_k E_k^{(2)}\left(\sum_j E_j^{(1)}\rho(E_j^{(1)})^\dagger\right)(E_k^{(2)})^\dagger \\ &= \sum_{j,k} E_k^{(2)}E_j^{(1)}\rho(E_j^{(1)})^\dagger(E_k^{(2)})^\dagger,\end{aligned}$$

so the Kraus operators are all products of the Kraus operators of $\mathcal{E}_1$ and $\mathcal{E}_2$, i.e. $\{E_k^{(2)}E_j^{(1)}\}$.

(b) Determine a Kraus representation for the channel Tr which maps $\rho \mapsto \operatorname{tr}\rho$ for a mixed quantum state ρ in d dimensions.

Answer: The channel has d Kraus operators, $E_k = \langle k|$:

$$\operatorname{Tr}(\rho) = \sum_k \langle k|\rho|k\rangle = \operatorname{tr}\rho.$$

(c) Let $\mathcal{E}$ and $\mathcal{F}$ be quantum channels with d Kraus operators each, E_k and F_k (respectively), such that for all j, $F_j = \sum_{k=1}^d U_{jk}E_k$ for some unitary matrix U. Show that $\mathcal{E}$ and $\mathcal{F}$ are actually the same quantum channel.

Answer: We have

$$\begin{aligned}\mathcal{F}(\rho) &= \sum_j F_j\rho F_j^\dagger = \sum_j\left(\sum_k U_{jk}E_k\right)\rho\left(\sum_\ell U_{j\ell}^* E_\ell^\dagger\right) \\ &= \sum_{k,\ell} E_k\rho E_\ell^\dagger \sum_j U_{jk}U_{j\ell}^* = \sum_k E_k\rho E_k^\dagger = \mathcal{E}(\rho),\end{aligned}$$

where we use unitarity of U.